Spectral Characterization of Trees

by

Faisal Abdul-Karim Fairag

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

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In

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Fairag, Faisal Abdul-karim, M.S.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1989

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under the direction of his thesis committee , and approved by all its members, has been presented to and accepted by the Dean, College of Graduate Studies, in partial fulfillment of the requirements for the degree of

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بسم الله الرحمن الرحيم

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الجذر المميز للمخطط هو الجذر المميز لمصفوفة التجاور لذلك المخطط . الرمز λ_1 يرمز لأكبر جذر مميز ، والرمز λ_2 يرمز لثاني أكبر جذر مميز .

في هذا البحث ندرس عائلة المخططات الشجريه التي تملك ثلاثة رؤوس منتهيه والتي تحقق مساواه معينه فيها λ_1 و λ_2 . القيمة العليا والدنيا لثاني أكبر جذر مميز λ_2 قد بُحثت ودُرست في هذا البحث .

درجة الماجستير في العلوم

جامعة الملك فهد للبترول والمعادن

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THESIS ABSTRACT

Full Name Of Student : **Faisal Abdul-karim Fairag**

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The eigenvalues of a graph are the eigenvalues of its adjacency matrix. Let λ_1 and λ_2 denote the largest and the second largest eigenvalue, respectively. The family of trees having only three end vertices and satisfying a certain equality involving λ_1 and λ_2 is characterized. Also, bounds on λ_2 for the above family of trees are investigated.

MASTER OF SCIENCE DEGREE

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

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Introduction

The effective use of spectra in graph theoretic investigations depends on our ability to take two major steps. First, we must be able to calculate the spectra of large classes of graphs or, more generally, we must be able to translate graph theoretic information into spectral information. Second, we must be able to deduce properties of graphs from their spectra properties. It is interesting that there is a large variety of methods for linking graphs with their spectra in both these senses.

The literature on the spectral radius λ_1 is rich and so much of it is classical. However, very little research has been done on the second largest eigenvalue λ_2. This thesis is aiming towards that goal for a family of graphs which we call $Y-trees$.

In Chapter 1 we put down all the basic properties of the spectrum of a graph that are needed later. We also give illustrations on four binary operations and their spectra. In Chapter 2 we calculate the spectra of those trees with a structure similar to paths and stars. We define and compute the eigenvalues of near-paths and near-stars. In the last chapter, we characterize the family of trees having only three end vertices and satisfying a certain equality involving λ_1 and λ_2. Finally we consider bounds on the second largest eigenvalue λ_2 for this family.

Chapter 1

Basic Properties of the Spectrum of a Graph

This introductory chapter is devoted mainly to definitions and basic concepts to make this thesis self contained. The well-known theorems about the spectra of graphs are given together with some examples and methods on computing the characteristic polynomial of a given graph. In the last section we consider four mutually related operations on graphs. These are the binary operations: the union, the join, the conjunction, and the Cartesian product of graphs. Several illustrations on trees are also presented.

1.1 Basic Definitions and Concepts

A graph $G = (V, E)$ consists of two sets of: a finite set V of elements called vertices and a finite set E of elements called edges. The vertices v_i and v_j associated with an edge e_ℓ are called <u>the end vertices</u> of e_ℓ. The edge e_ℓ is then devoted as

$e_\ell = (v_i, v_j)$. If $e_\ell = (v_i, v_i)$, then the edges e_ℓ is called a self loop at vertex v_i. All edges having the same pair of end vertices are called parallel edges. A graph is called a simple graph if it has no parallel edges or self-loops. A graph is of order n if its vertex set has n elements. Two vertices are adjacent if they are the end vertices of some edges. An edge is said to be incident on its end vertices. The number of edges incident on a vertex v_i is called the degree of the vertex, and it is denoted by $d(v_i)$. A vertex of degree 1 is called an end vertex. A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if V' and E' are respectively, subsets of V and E such that an edge (v_i, v_j) is in E' only if v_i and v_j are in V'.

A trail of a graph $G = (V, E)$ is a finite alternating sequence of vertices and distinct edges

$$v_0, e_1, v_1, e_2, \cdots, v_{k-1}, e_k, v_k$$

beginning and ending with vertices such that v_{i-1} and v_i are the end vertices of the edge $e_i, 1 \leq i \leq k$. A trail is open if its end vertices are distinct ; otherwise it is closed. An open trail is a path if all its vertices are disinct. A closed trail is a circuit if all its vertices except the end vertices are distinct. A graph G is connected if there exists a path between every pair of vertices in G. If v_i is a vertex of a graph $G = (V, E)$, then the graph $\underline{G - v_i} = (V', E')$ is the graph obtained after removing from G the vertex v_i and all the edges incident to v_i. If e_i is an edge of a graph $G = (V, E)$, then $\underline{G - e_i}$ is the subgroup of G obtained after removing from G the edge e_i. A complete graph G is a simple graph in which every pair of vertices is adjacent. If a complete graph G has $\underline{n - vertices}$ then it will be denoted by K_n. A graph $G = (V, E)$ is bipartite if its vertex set V can be partitioned into two subsets V_1 and V_2 such that each edge of E has one end vertex in V_1 and another in V_2.

A tree is a connected graph and has no circuits.

The adjacency matrix A of a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$ is the $n \times n$ matrix $[a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & if \quad (v_i, v_j) \in E(G) \\ 0 & otherwise \end{cases}$$

The characteristic polydnomial of G is defined to be the characteristic polynomial of the adjacency matrix, i.e. characteristics polynomial of $G = \phi(G, \lambda) = |A - \lambda I|$. The eigenvalues of graph G of order n are defined to be the roots of the characteristic polynomial of G. Since $A(G)$ is real symmetric matrix, the eigenvalues must be real, and may be ordered

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n,$$

the sequence of n eigenvalues is called the spectrum of G . Let G be a graph of order n. If the distinct eigenvalues of G are

$$\lambda_0 > \lambda_1 > \cdots > \lambda_{s-1},$$

and their multiplicities are

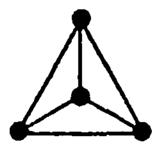
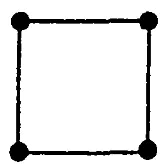
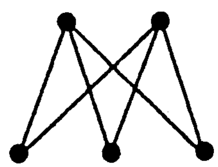
$$m(\lambda_0), m(\lambda_1), \cdots, m(\lambda_{s-1})$$

then we shall write

$$Spec\ G = \begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \cdots & m(\lambda_{s-1}) \end{pmatrix}$$

We illustrate the above definitions by the examples given on the next page.

EXAMPLES

	Graph	Adjacency matrix	Characteristic polynomial	Eigenvalues
K_2:		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	x^2-1	$1, -1$
P_3:		$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	x^3-2x	$\sqrt{2}, -\sqrt{2}, 0$
K_4:		$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$	x^4-6x^2-8x-3	$3, -1, -1, -1$
C_4:		$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$	x^4-4x^2	$2, -2, 0, 0$
$K_{2,3}$:		$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$	x^5-6x^3	$\sqrt{6}, -\sqrt{6}, 0, 0, 0$

1.2 Well Known Results

For the sake of completeness we mention below some of the relevant results on graph spectra, computation of the characteristic polynomial, and bounds on eigenvalues. Moreover, we give examples that illustrate the theorems.

Theorem 1.2.1 *If G is a disconnected graph, then the spectrum of G is the union of the spectra of the components of G.*

Theorem 1.2.2 *If ψ is any polynomial, and if λ is any eigenvalue of the matrix A, then $\psi(\lambda)$ is an eigenvalue of the matrix $\psi(A)$.*

Theorem 1.2.3 *The sum of the eigenvalues of a graph is zero.*

Theorem 1.2.4 *If G is a connected graph with m distinct eigenvalues and with diameter d, then $m > d$.*

Theorem 1.2.5 *[17] A graph has exactly one positive eigenvalue if and only if the non-isolated vertices form a complete multipartite graph.*

Theorem 1.2.6 *[17] Let G be a tree with S vertices of degree one and t vertices adjacent to vertices of degree one. Then the multiplicity of any eigenvalue (non-zero eigenvalue) is less than $S(t$, for $t \neq 1)$.*

Computing The Characteristic Polynomial

Theorem 1.2.7 *[15] The coefficients a_i of $\phi(G;\lambda) = \sum_i^n a_i\lambda^{n-i}$ are given by*

$$a_i = \sum_H (-1)^{K(H)} 2^{C(H)}$$

where the summation extends over all subgraphs H of G on i vertices whose components are either single edge or circuits, and where $K(H)$ and $C(H)$ denote, respectively, the number of components and circuits in H.

Theorem 1.2.8 *[16] Let v be a vertex of a graph G, let $\mathcal{G}(V)$ be the collection of circuits conntaining v and let $V(Z)$ denote the set of vertices in the circuit Z. Then the characteristics polynomial $\phi(G;\lambda)$ satisfies*

$$\phi(G;\lambda) = \lambda\phi(G-v;\lambda) - \sum_w \phi(G-v-w;\lambda) - 2\sum_Z \phi(G-V(Z);\lambda)$$

where the first summation extends over those vertices w adjacent to v, and the second summation extends over all $Z \in \mathcal{G}(v)$.

Corollary 1.2.1 *If v is an end-vertex of a graph G, and if w is the vertex adjacent to v, then*

$$\phi(G;\lambda) = \lambda\phi(G-v;\lambda) - \phi(G-v-w;\lambda)$$

Theorem 1.2.9 *[16] Let $e = vw$ be an edge of G, and let $\mathcal{G}(e)$ be the set of circuits containing e. Then $\phi(G;\lambda)$ satisfies*

$$\phi(G;\lambda) = \phi(G-e;\lambda) - \phi(G-v-w;\lambda) - 2\sum_Z \phi(G-V(Z);\lambda)$$

where the summation extends over all $Z \in \mathcal{G}(e)$.

Theorem 1.2.10 *[11] If u is a vertex of a tree T, then*

$$\phi(T;\lambda) = \lambda\phi(T-u;\lambda) - \sum_{v\sim u}\phi(T-u-v;\lambda)$$

$u \sim v$ means u and v are adjacent.

Bound On Eigenvalues

Theorem 1.2.11 *[14][5] If G is a connected graph with at least two vertices, then*

(i) Its largest eigenvalue λ_1 is a simple root of $\phi(G;\lambda)$;

(ii) Corresponding to the eigenvalue λ_1 , there is an eigenvector x_1 all of whose coordinates are positive;

(iii) If λ is any other eigenvalue of G , then $-\lambda_1 \leq \lambda \leq \lambda_1$;

(iv) the deletion of any edge of G decreases the largest eigenvalue.

Theorem 1.2.12 (6) *The following statements are equivalent for a connected graph G.*

(i) G is a bipartite graph;

(ii) $\lambda_p = -\lambda_1$;

(iii) $\lambda_i = -\lambda_{p+1-i}$ for $1 \leq i \leq \frac{1}{2}(P-1)$

(iv) $a_{2i-1} = 0$ for $1 \leq i \leq \frac{1}{2}(P+1)$

(v) $\sum_{j=1}^{P} \lambda_j^{2i-1} = 0$ *for all* $i \geq 1$

(vi) $\sum_{j=1}^{P} \lambda_j^{2[\frac{1}{2}(P+1)]-1} = 0$

Theorem 1.2.13 (4) *If G is a connected graph of order n, then*

$$2\cos(\frac{\pi}{n+1}) \leq \lambda_1 \leq n-1$$

The lower bound occurs only when G is the path P_n, and the upper bound occurs only when G is the complete graph K_n.

Theorem 1.2.14 *[17] If G is a connected graph with $\lambda_1 = 2$, then G is either $K_{1,4}$, a circuit graph C_n, or one of the trees in Figure 1. Moreover, if G is a connected graph wth $\lambda_1 < 2$, then G is a subgraph of one of these graphs.*

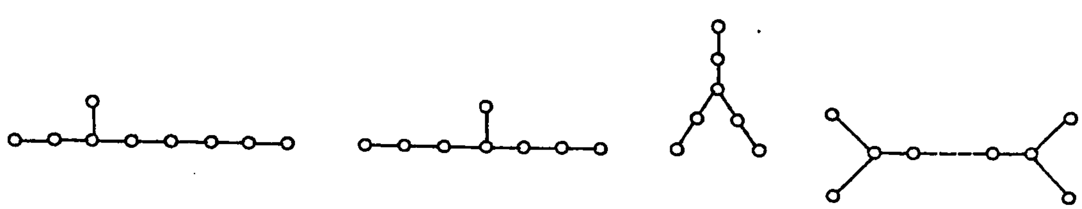


Figure 1

Theorem 1.2.15 *Let G be a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let the spectrum of $G - v_1$ be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$. Then the spectrum of $G - v_1$ is "interlaced" with the spectrum of G, that is,*

$$\lambda_1 \geq u_1 \geq \lambda_2 \geq u_2 \geq \cdots \geq u_{n-1} \geq \lambda_n$$

1.3 Operations On Graphs And The Resulting Spectra

In this section we give several definitions and illustrations on the operations on graphs together with the spectra of these operations. We shall assume in this section that G_1 and G_2 are two graphs with disjoint vertex sets.

The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. An example is shown in Figure 2(c).

Theorem 1.3.1 (3) *The spectrum of the union is given by*

$$S(G_1 \cup G_2) = S(G_1) \cup S(G_2).$$

In figure 2(c),

$$\begin{aligned} S(G_1 \cup G_2) &= \{\sqrt{3}\ , \sqrt{3}\ , 0, 0, 0, 0, -\sqrt{3}\ , -\sqrt{3}\ \} \\ &= \{\sqrt{3}\ , 0, 0, -\sqrt{3}\} \cup \{-\sqrt{3}\ , 0, 0, -\sqrt{3}\ \} \\ &= S(G_1) \cup S(G_2). \end{aligned}$$

The Join $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{(u,v) : u \in V(G_1) \ \ \text{and} \ \ v \in V(G_2)\}.$$

An example is shown in figure 2(d).

The conjunction $G = G_1 \wedge G_2$ *has* $V(G) = V(G_1) \times V(G_2)$ *and*

$$E(G) = \{((u_1, v_1), (u_2, v_2)) : (u_1, u_2) \in E(G_1), (v_1, v_2) \in E(G_2)\}.$$

Two examples are shown in figure 2(e) and figure 3.

Theorem 1.3.2 (3) *The spectrum of the conjunction is given by*

$$S(G_1 \wedge G_2) = S(G_1) \cdot S(G_2)$$

that is, the product of each number in $S(G_1)$ *with every number in* $S(G_2)$.

In figure 1(e),

$$\begin{aligned} S(G_1 \wedge G_2) &= \{3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, -3\} \\ &= \{\sqrt{3}\ , 0, 0, -\sqrt{3}\ \} \cdot \{\sqrt{3}\ , 0, 0, -\sqrt{3}\ \} \\ &= S(G_1) \cdot S(G_2). \end{aligned}$$

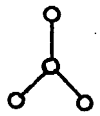
The cartesian product $G = G_1 \times G_2$ *has* $V(G) = V(G_1) \times V(G_2)$ *and* $E(G) = \{((u_1, u_2), (v_1, v_2))$: *either* $u_1 = v_1$ *and* $(u_2, v_2) \in E(G_2)$ *or* $u_2 = v_2$ *and* $(u_1, v_1) \in E(G_1)\}$. *An example is shown in Figure 2(e).*

Theorem 1.3.3 (3) *The spectrum of the Cartesian Product is given by*

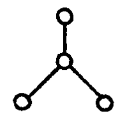
$$S(G_1 \times G_2) = S(G_1) + S(G_2),$$

which is the sum of each number in $S(G_1)$ *with every number in* $S(G_2)$*. In Figure 2(f),*

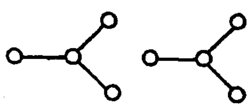
$$\begin{aligned} S(G_1 \times G_2) &= \{2\sqrt{3}\ ,\sqrt{3}\ ,\sqrt{3}\ ,\sqrt{3}\ ,\sqrt{3}\ ,0,0,0,0,0,0, \\ -\sqrt{3} &\ ,-\sqrt{3}\ ,-\sqrt{3}\ ,-\sqrt{3}\ ,-2\sqrt{3}\ \} \\ &= \{\sqrt{3}\ ,0,0,-\sqrt{3}\ \} + \{\sqrt{3}\ ,0,0,-\sqrt{3}\ \} \\ &= S(G_1) + S(G_2). \end{aligned}$$



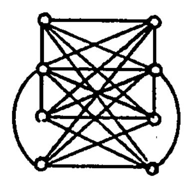
(a) G_1



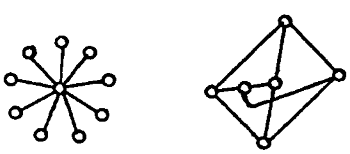
(b) G_2



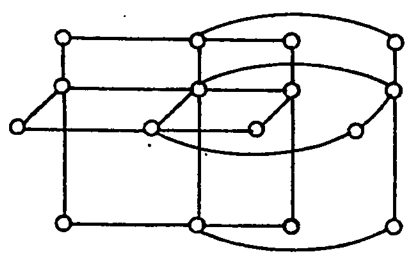
(c) $G_1 \cup G_2$



(d) $G_1 + G_2$



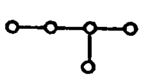
(e) $G_1 \wedge G_2$



(f) $G_1 \times G_2$

FIGURE 2

G_1



G_2

$G_1 \wedge G_2$

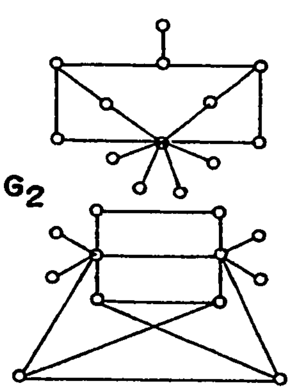


FIGURE 3

Chapter 2

Paths, Stars and Other Trees

In this chapter we recall some theorems [2,12,13] about arbitrary trees and calculate the spectra of those trees with a structure similar to paths and stars. In particular, we will compute the eigenvalues of near-paths and near-stars. Table 1 is provided to give the reader a cursory glimpse about all trees of order 7.

2.1 Preliminary Results And Examples

The first theorem is classical in the sense that it is deducible from the Perron-Frobenius algebra of matrices. The upper and lower bounds on λ_1 *in theorem (2.1.2) is due to Lovasz and Pelikan [12]. Howeer, theorems (2.1.3) and (2.1.4) are very recent results giving an upper bound for the second largest eigenvalue of an arbitrary tree.*

Theorem 2.1.1 *The following statements are equivalent for a graph* T*:*

(i) T *is a tree;*

(ii) $\lambda_n = -\lambda_1$ *;*

(iii) $\lambda_i = -\lambda_{n+1-i}$ *for* $1 \leq i \leq \frac{1}{2}(n-1)$

(iv) $a_{2i-1} = 0$ *for* $1 \leq i \leq \frac{1}{2}(n+1)$ *where* $a_{2i-1} =$ *The coefficient of* x^{2i-1}

(v) $\sum_{j=1}^{n} \lambda_j^{2i-1} = 0$ *for all* $i \geq 1$

(vi) $\sum_{j=1}^{n} \lambda_j^{2[\frac{1}{2}(n+1)]-1} = 0$

Theorem 2.1.2 *If* T *is a tree of order* n*, then:*

$$2\cos(\frac{\pi}{n+1}) \leq \lambda_1 \leq \sqrt{n-1}$$

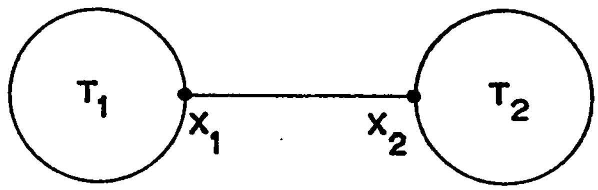
The lower bound occurs only when T *is the path* P_n*, and the upper bound occurs only when* T *is the star* S_n*.*

Remark 2.1.1 *Note that* λ_1 *in Table 1 is arranged in an increasing order beginning with path and ending with the star.*

Theorem 2.1.3 *Let* T *be a tree with* $\lambda_2(T) \leq k$*. Then either*

(1) T *contains a vertex* x *such that* $\lambda_1(T-x) \leq k$ *or*

(2) T is a k-twin (i.e.) T has the shape



with subtrees T_1 and T_2 satisfying

$$\lambda_1(T_i - x_i) < k < \lambda_1(T_i) \quad \textit{for} \quad i = 1, 2.$$

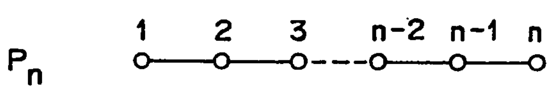
Theorem 2.1.4 *If T is a tree of order n then*

$$\lambda_2(T) \leq \sqrt{\frac{n-3}{2}}.$$

The examples shown below will play a central *role in the investigations ahead.*

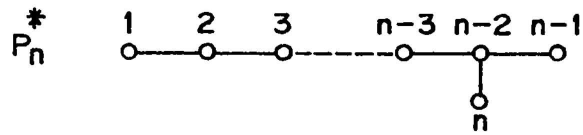
(1) Path P_n

The eigenvalues of P_n are $2\cos(\frac{k\pi}{n+1}), k = 1, 2, \cdots, n$.



(2) Near-Path P_n^*

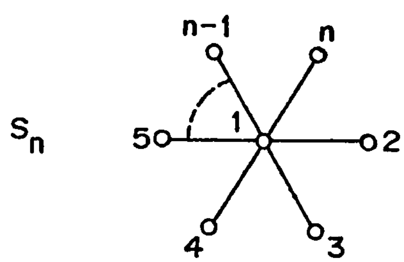
The eigenvalues of P_n^* are $0, \mp 2\cos\frac{(2k+1)\pi}{2n-2}, k = 0, 1, 2, \cdots, n-2.$



(3) Star S_n

The eigenvalues of S_n are

$$\sqrt{n-1}, \underbrace{0, 0, 0, \cdots, 0, 0, 0,}_{n-2} -\sqrt{n-1}$$



(4) Near-Star S_n^*

The eigenvalues of S_n^* are

$$\sqrt{\frac{(n-1)+\sqrt{n^2-6n+13}}{2}}, \sqrt{\frac{(n-1)-\sqrt{n^2-6n+13}}{2}}, \underbrace{0, \cdots, 0,}_{n-4} -\sqrt{\frac{(n-1)-\sqrt{n^2-6n+13}}{2}}, -\sqrt{\frac{(n-1)+\sqrt{n^2-6n+13}}{2}}$$

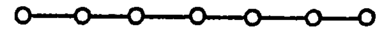
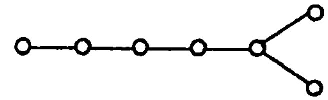
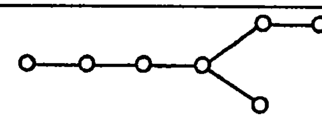
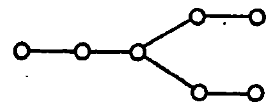
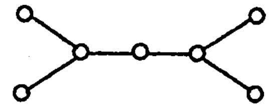
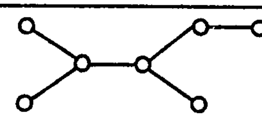
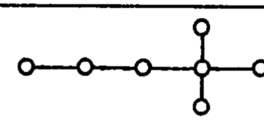
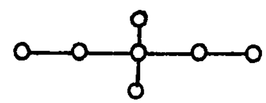
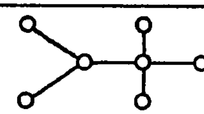
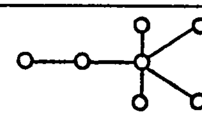
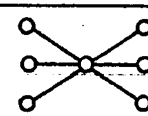
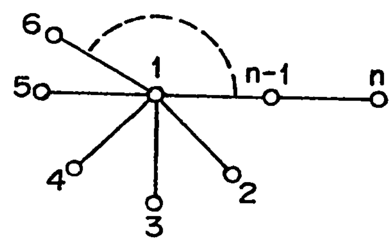
λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	Tree	
1.84776	1.41421	0.765367	0	-0.765367	-1.41421	-1.84776		(1)
1.93185	1.41421	0.517638	0	-0.517638	-1.41421	-1.93185		(2)
1.96962	1.28558	0.68404	0	-0.68404	-1.28558	-1.96962		(3)
2	1	1	0	-1	-1	-2		(4)
2	1.41421	0	0	0	-1.41421	-2		(5)
2.05288	1.20864	0.569973	0	-0.569973	-1.20864	-2.05288		(6)
2.101	1.25928	0	0	0	-1.25928	-2.101		(7)
2.13578	1	0.662153	0	-0.662153	-1	-2.13576		(8)
2.17533	1.12603	0	0	0	-1.12603	-2.17533		(9)
2.28825	0.874032	0	0	0	-0.874032	-2.28825		(10)
2.44949	0	0	0	0	0	-2.44949		(11)

Table 1: All Trees of order seven and their eigenvalues

S_n^*



2.2 The Near Path P_n^*

The next two lemmas will pave the way for the proof of theorem (2.2.1) which will provide a formula for the calculation of the spectrum of our near-path P_n^*.

Lemma 2.2.1 *Let $\{x_n\}$ be a sequence defined by the recursion*

$$x_n = ax_{n-1} + bx_{n-2}, a^2 + 4b \neq 0$$

then $x_n = c_1 y_1^n + c_2 y_2^n$ where

$$y_{1,2} = \frac{a \pm \sqrt{a^2 + 4b}}{2}$$

and c_1, c_2 are determined by the initial values of $\{x_n\}$.

Lemma 2.2.2 *Let* $(\frac{x}{a})^n = 1$ *where* a *is complex number. Then the solutions of this equation [3] are*

$$x = \epsilon \cdot a \quad \textit{where} \quad \epsilon = e^{\frac{2k\pi}{n}}, k = 0, 1, 2, \cdots, n-1$$

Let P_n^* *be the tree of order* n *having three end vertices with the shape as shown in Figure(4).*

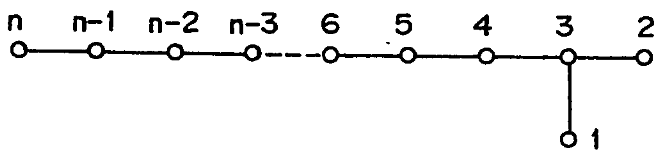


Figure 4. A near-path P_n^*

Theorem 2.2.1 *The eigenvalues of* P_n^* *are*

$$0, \pm 2\cos\frac{(2k+1)\pi}{2n-2}, k = 0, 1, \cdots, n-2$$

and the largest eigenvalue of P_n^* *is* $2\cos(\frac{\pi}{2n-2})$.

Proof: *Let* P_n^* *be a tree of order* n *with the shape as shown in Figure 1. By corollary (1.2.1), we have*

$$\phi(P_n^*; \lambda) = \lambda \cdot \phi(P_{n-1}^*; \lambda) - \phi(P_{n-2}^*; \lambda)$$

which can be considered to be a recursive definition of $\phi(P_n^*;\lambda)$. By lemma (2.2.1), we have $a=\lambda, b=-1, a^2+4b=\lambda^2-4\neq 0$. Hence

$$\phi(P_n^*;\lambda)=c_1y_1^n+c_2y_2^n \tag{2.1}$$

where

$$y_1=\frac{\lambda+\sqrt{\lambda^2-4}}{2},\ y_2=\frac{\lambda-\sqrt{\lambda^2-4}}{2}$$

To find c_1 and c_2, we use the initial values of the sequence. We have $\phi(P_4^*;\lambda)=\lambda^4-3\lambda^2$ and $\phi(P_5^*;\lambda)=\lambda^5-4\lambda^3+2\lambda$. Use (2.1) with $n=4,5$ gives

$$c_1y_1^4+c_2y_2^4=\phi(P_4^*;\lambda)=\lambda^4-3\lambda^2 \tag{2.2}$$

$$c_1y_1^5+c_2y_2^5=\phi(P_5^*;\lambda)=\lambda^5-4\lambda^3+2\lambda \tag{2.3}$$

The system of equations (2.2) and (2.3) is in two unknowns c_1 and c_2. Using Cramer's rule gives

$$c_1=\frac{\begin{vmatrix}\lambda^4-3\lambda^2 & y_2^4\\ \lambda^5-4\lambda^3+2\lambda & y_2^5\end{vmatrix}}{\begin{vmatrix}y_1^4 & y_2^4\\ y_1^5 & y_2^5\end{vmatrix}} \quad \textit{and} \quad c_2=\frac{\begin{vmatrix}y_1^4 & \lambda^4-3\lambda^2\\ y_1^5 & \lambda^5-4\lambda^3+2\lambda\end{vmatrix}}{\begin{vmatrix}y_1^4 & y_2^4\\ y_1^5 & y_2^5\end{vmatrix}}$$

We have

$$\begin{vmatrix}y_1^4 & y_2^4\\ y_1^5 & y_2^5\end{vmatrix}=-\sqrt{\lambda^2-4}$$

$$\begin{vmatrix}\lambda^4-3\lambda^2 & y_2^4\\ \lambda^5-4\lambda^3+2\lambda & y_2^5\end{vmatrix}=-y_2\cdot\lambda\cdot\sqrt{\lambda^2-4}$$

$$\begin{vmatrix}y_1^4 & \lambda^4-3\lambda^2\\ y_1^5 & \lambda^5-4\lambda^3+2\lambda\end{vmatrix}=-y_1\cdot\lambda\cdot\sqrt{\lambda^2-4}$$

Hence, $c_1=\lambda\cdot y_2$ and $c_2=\lambda y_1$. Substituting these values into the equation (2.1) gives

$$\phi(P_n^*;\lambda) = (\lambda y_2)y_1^n+(\lambda y_1)y_2^n$$

$$= \lambda \cdot y_1^{n-1} + \lambda \cdot y_2^{n-1}$$

$$= \lambda \left[y_1^{n-1} + y_2^{n-1} \right]$$

Thus, the eigenvalues of P_n^* satisfy

$$y_1^{n-1} = -y_2^{n-1} \qquad or \qquad \lambda = 0$$

To solve $y_1^{n-1} = -y_2^{n-1} = (e^{\frac{\pi i}{n-1}} y_2)^{n-1}$, use the lemma (2.2.2). We have

$$y_1 = \epsilon \cdot e^{\frac{\pi i}{n-1}} y_2 \text{ where } \epsilon = e^{\frac{2k\pi i}{n-1}}, k = 0, 1, \cdots, n-2 .$$

Multiply both sides by $\lambda + \sqrt{\lambda^2 - 4}$, and then solve for λ^2, giving

$$\lambda^2 = \frac{\left[e^{\frac{-\pi i}{2(n-1)}} \cdot (1 + \epsilon e^{\frac{\pi i}{n-1}}) \right]^2}{\epsilon}.$$

λ being real implies

$$\begin{aligned} \lambda &= \pm \left| e^{\frac{-\pi i}{2(n-1)}} \cdot (1 + \epsilon e^{\frac{\pi i}{n-1}}) \right| \\ &= \pm \left| e^{\frac{-\pi i}{2(n-1)}} \right| \cdot \left| (1 + \epsilon e^{\frac{\pi i}{n-1}}) \right| \end{aligned}$$

$$\begin{aligned} \lambda &= \pm \left| 1 + \epsilon e^{\frac{\pi i}{n-1}} \right| \\ &= \pm \left| 1 + e^{\frac{2k\pi i}{n-1}} \cdot e^{\frac{\pi i}{n-1}} \right| \text{ where } k = 0, 1, \cdots, n-2 \\ &= \pm \left| 1 + e^{\frac{(2k+1)\pi i}{n-1}} \right| \text{ where } k = 0, 1, \cdots, n-2 \\ &= \pm \left| 1 + \cos \frac{(2k+1)\pi}{n-1} + i \sin \frac{(2k+1)\pi}{n-1} \right| \\ &= \pm \sqrt{(1 + \cos \frac{(2k+1)\pi}{n-1})^2 + (\sin \frac{(2k+1)\pi}{n-1})^2} \end{aligned}$$

$$= \pm\sqrt{2+2\cos\frac{(2k+1)\pi}{n-1}}$$
$$= \pm\sqrt{4\cos^2\frac{(2k+1)\pi}{2n-2}}$$
$$= \pm 2\cos\frac{(2k+1)\pi}{2n-2} \quad \textit{where } k=0,1,\cdots,n-2$$

Therefore, the eignevalues of P_n^* *are* $0, \pm 2\cos\frac{(2k+1)\pi}{2n-2}$ *where* $k = 0,1,\cdots,n-2$ *and the largest eigenvalue is* $2\cos(\frac{\pi}{2n-2})$.

2.3 Stars And Near-Stars

It is known [2] that the eigenvalues of the star S_n *are* $\sqrt{n-1}, 0, \cdots, 0, -\sqrt{n-1}$. *Now we develop several formulas to help us calculate the eigenvalues of some trees which have a structure similar to that of* S_n.

(1) The Near Star S_n^*

Let S_n^* *be a tree of order n with the shape as shown in Figure 5. Then the eigenvalues of* S_n^* *together with their multiplicities are*

$$\begin{pmatrix} \sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} & \sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & 0 & -\sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & -\sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} \\ 1 & 1 & n-4 & 1 & 1 \end{pmatrix}$$

where $\Delta = (n-1)^2 - 4(n-3)$.

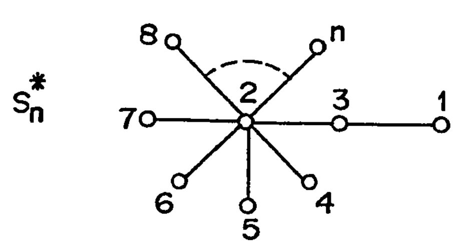


Figure 5: A Near-Star S_n^*

Proof : *Let S_n^* be a tree of order n and let the vertex 1 be as shown in Figure 5. Corollary (1.2.1) gives*

$$\phi(S_n^*;\lambda) = \lambda\phi(S_{n-1};\lambda) - \phi(S_{n-2};\lambda)$$

Since $\phi(S_n;\lambda) = \lambda^{n-2}(\lambda^2 - (n-1))$, *we get*

$$\begin{aligned}\phi(S_n^*;\lambda) &= \lambda\{\lambda^{n-3}(\lambda^2-(n-2))\} - \{\lambda^{n-4}(\lambda^2-(n-3))\}\\ &= \lambda^{n-4}\{\lambda^4-(n-1)\lambda^2+(n-3)\}\end{aligned}$$

Hence , $\lambda^{n-4} = 0$ *or* $\lambda^4 - (n-1)\lambda^2 + (n-3) = 0$.

Therefore the eigenvalues of S_n^* are

$$\begin{pmatrix} \sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} & \sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & 0 & -\sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & -\sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} \\ 1 & 1 & n-4 & 1 & 1 \end{pmatrix}$$

where $\Delta = (n-1)^2 - 4(n-3)$.

(2) The Tree S_n^{**}

Let S_n^{**} be a tree order n with the shape as shown in Figure 6. Then the eigenvalues of S_n^{**} are

$$\begin{pmatrix} \sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} & \sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & 0 & -\sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & -\sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} \\ 1 & 1 & n-4 & 1 & 1 \end{pmatrix}$$

where $\Delta = (n-1)^2 - 4(2n-7)$.

S_n^{**} :

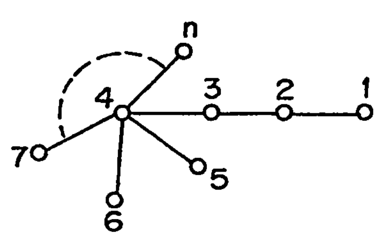


Figure 6: A tree S_n^{**}

Proof : *Let S_n^{**} be a tree of order n and let 1 and 2 be two vertices in S_n^{**} as shown in Figure 6. Corollary (1.2.1) gives*

$$\phi(S_n^{**};\lambda) = \lambda\phi(S_{n-1}^{*};\lambda) - \phi(S_{n-2};\lambda)$$

Since $\phi(S_n;\lambda) = \lambda^{n-2}(\lambda^2-(n-1))$, *and* $\phi(S_n^{*};\lambda) = \lambda^{n-4}\{\lambda^4-(n-1)\lambda^2+(n-3)\}$, *we get*

$$\phi(S_n^{**};\lambda) = \lambda^{n-4}\{\lambda^4-(n-1)\lambda^2+(2n-7)\}$$

Hence , $\lambda^{n-4} = 0$ *or* $\lambda^4-(n-1)\lambda^2+(n-3) = 0$.

Therefore the eigenvalues of S_n^{**} are

$$\begin{pmatrix} \sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} & \sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & 0 & -\sqrt{\frac{(n-1)-\sqrt{\Delta}}{2}} & -\sqrt{\frac{(n-1)+\sqrt{\Delta}}{2}} \\ 1 & 1 & n-4 & 1 & 1 \end{pmatrix}$$

where $\Delta = (n-1)^2 - 4(2n-7)$.

(3) The Constellation C

Let C be a tree of order $kn+1$ with the shape as shown in Figure 7. Then the eigenvalues of C are

$$\begin{pmatrix} \sqrt{\frac{(n+k-1)+\sqrt{\Delta}}{2}} & \sqrt{\frac{(n+k-1)-\sqrt{\Delta}}{2}} & \sqrt{n-1} & 0 & -\sqrt{n-1} & -\sqrt{\frac{(n+k-1)-\sqrt{\Delta}}{2}} & -\sqrt{\frac{(n+k-1)+\sqrt{\Delta}}{2}} \\ 1 & 1 & k-1 & k(n-2)-1 & k-1 & 1 & 1 \end{pmatrix}$$

where $\Delta = (n+k-1)^2 - 4(n-2)\cdot k$, $\quad n \geq 3$, $k \geq 1$.

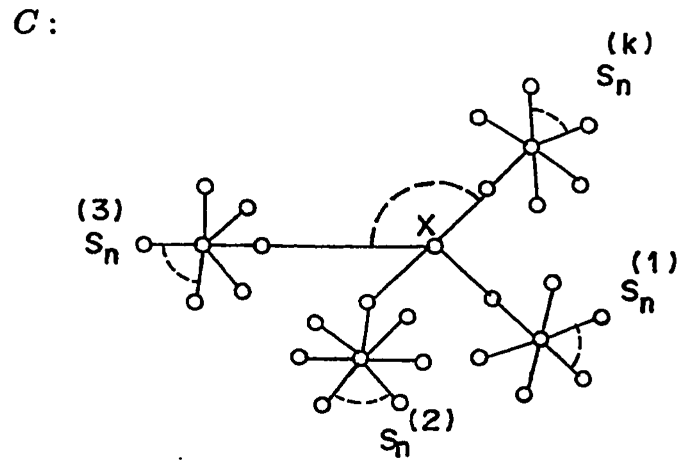


Figure 7: A tree of order $kn + 1$

Proof : *Let C be a tree of order $kn+1$ with the shape as shown in Figure 7 and let x be a vertex in C as shown in Figure 7. Theorem (1.2.10) gives*

$$\begin{aligned}\phi(C;\lambda) &= \lambda\cdot\phi^{k}(S_n;\lambda) - k\cdot\phi(S_{n-1};\lambda)\cdot\phi^{k-1}(S_n;\lambda)\\ &= \phi^{k-1}(S_n;\lambda)\{\lambda\phi(S_n;\lambda) - k\phi(S_{n-1};\lambda)\}\end{aligned}$$

Then $\phi^{k-1}(S_n;\lambda) = 0$ *or* $\lambda\phi(S_n;\lambda) - k\phi(S_{n-1};\lambda) = 0$.

$$\begin{aligned}\lambda\phi(S_n;\lambda) - k\phi(S_{n-1};\lambda) &= \lambda^{n-1}(\lambda^2 - (n-1)) - k\cdot\lambda^{n-3}(\lambda^2 - (n-2))\\ &= \lambda^{n-3}[\lambda^4 - (n+k-1)\ \lambda^2 + (n-2)k]\end{aligned}$$

Hence, the eigenvalues of C are

$$\begin{pmatrix} \sqrt{\frac{(n+k-1)+\sqrt{\Delta}}{2}} & \sqrt{\frac{(n+k-1)-\sqrt{\Delta}}{2}} & \sqrt{n-1} & 0 & -\sqrt{n-1} & -\sqrt{\frac{(n+k-1)-\sqrt{\Delta}}{2}} & -\sqrt{\frac{(n+k-1)+\sqrt{\Delta}}{2}} \\ 1 & 1 & k\text{-}1 & k(n\text{-}2)\text{-}1 & k\text{-}1 & 1 & 1 \end{pmatrix}$$

where $\Delta = (n+k-1)^2 - 4(n-2)k$.

(4) The Tree C^*

Let C^* be a tree of order $kn+1$ with the shape as shown in Figure 8. Then the eigenvalues of C^* are

$$\begin{pmatrix} \sqrt{(k+n-1)} & \sqrt{n-1} & 0 & -\sqrt{n-1} & -\sqrt{(k+n-1)} \\ 1 & k\text{-}1 & k(n\text{-}2)+1 & k\text{-}1 & 1 \end{pmatrix}$$

where $k \geq 1, n \geq 1$.

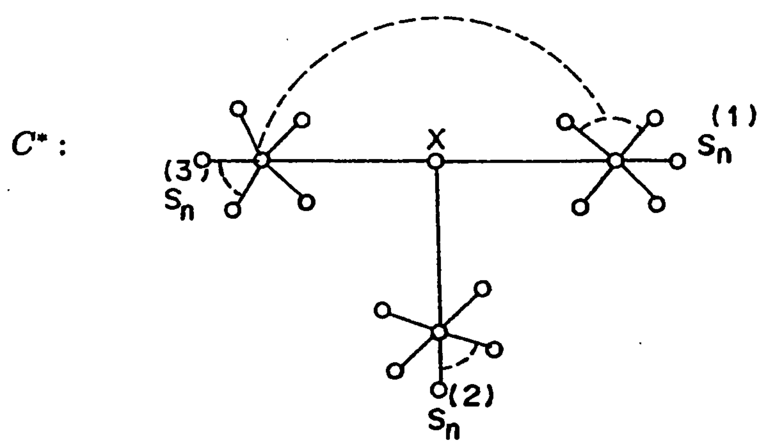


Figure 8: A tree of order $kn+1$

Proof : *Let C^* be a tree of order $kn+1$ with the shape as shown in Figure 8 and let x be a vertex in C^* as shown in Figure 8. Theorems (1.2.10) gives*

$$\begin{aligned} \phi(C^*;\lambda) &= \lambda \cdot \phi^k(S_n;\lambda) - k \cdot \lambda^{n-1} \cdot \phi^{k-1}(S_n;\lambda) \\ &= \phi^{k-1}(S_n;\lambda)\{\lambda^{n-1}(\lambda^2-(n-1)) - k\lambda^{n-1}\} \\ &= \lambda^{n-1} \cdot \phi^{k-1}(S_n;\lambda) \cdot \{\lambda^2 - (n+k-1)\} \end{aligned}$$

Therefore the eigenvalues of C^* are

$$\begin{pmatrix} \sqrt{(k+n-1)} & \sqrt{n-1} & 0 & -\sqrt{n-1} & -\sqrt{(k+n-1)} \\ 1 & k\text{-}1 & k(n\text{-}2)+1 & k\text{-}1 & 1 \end{pmatrix}$$

Remark 2.3.1 *From the tree C^* in (4) with $n = 2$ we see that the eigenvalues of the tree in Figure 9 are:*

$$\begin{pmatrix} \sqrt{(k+1)} & 1 & 0 & -1 & -\sqrt{k+1} \\ 1 & k\text{-}1 & 1 & k\text{-}1 & 1 \end{pmatrix}$$

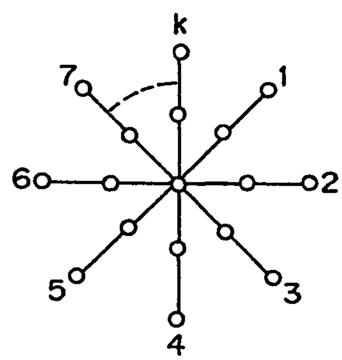


Figure 9: A tree

Remark 2.3.2 *From the tree C in (3) with $n = 3$, we obtain the spectrum as in Figure 10.*

$$\begin{pmatrix} \sqrt{\frac{k+2+\sqrt{k^2+2k}}{2}} & \sqrt{\frac{k+2-\sqrt{k^2+2k}}{2}} & \sqrt{2} & 0 & -\sqrt{2} & -\sqrt{\frac{k+2-\sqrt{k^2+2k}}{2}} & -\sqrt{\frac{k+2+\sqrt{k^2+2k}}{2}} \\ 1 & 1 & k\text{-}1 & k\text{-}1 & k\text{-}1 & 1 & 1 \end{pmatrix}$$

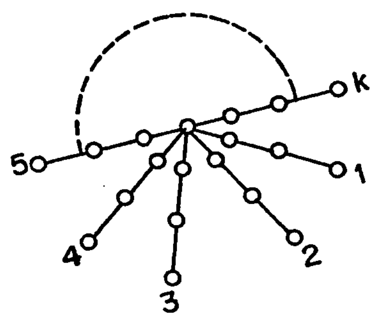


Figure 10: A tree

Chapter 3

The Eigenvalues Of A Family Of Trees

Let G be a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and let the spectrum of $G - v$ be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$. Then the spectrum jof $G - v$ is "interlaced" with the spectrum of G where v is a vertex in G, i.e.,

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n.$$

Three questions of inerest will be considered in this chapter.

(1) Find a vertex v which minimize $\lambda_1(G - v)$, and

(2) characterize those vertices v which satisfy $\lambda_1(G - v) = \lambda_2(G)$.

(3) Give bounds on the second largest eigenvalue for the case of a tree having exactly three end vertices

3.1 The Second Largest Eigenvalue

A tree with three end vertices is called a $\underline{Y-tree}$. *In a* $Y-$*tree, there is only one vertex having degree 3. We denote it by* u. *Removing this vertex* u *gives three paths of order* k_1, k_2 *and* k_3 *respectively, where* $k_1 \geq k_2 \geq k_3 > 0$. *So we denote a* $Y-$*tree by these three numbers as* $Y = Y(k_1, k_2, k_3)$. *Two* $Y-$ *trees* $Y_1 = (k_1, k_2, k_3)$ *and* $Y_2 = (m_1, m_2, m_3)$ *are isomorphic if and only if* $k_1 = m_1, k_2 = m_2$ *and* $k_3 = m_3$.

In what follows we will use the theory of integer partitioning [7] to count the number of non-isomorphic $Y-$ *trees of order* n.

A <u>*partition*</u> *of a positive integer* n *is a representation of* n *as a sum of positive integers.* $\underline{P_k(n)}$ *is the number of (unordered) partitions of* n *into* k *parts. We have*

$$
\begin{aligned}
P_3(n) &= n^2/12, & n \equiv 0(6)\\
&= n^2/12 - 1/12, & n \equiv 1(6)\\
&= n^2/12 - 1/3, & n \equiv 2(6)\\
&= n^2/12 + 1/4, & n \equiv 3(6)\\
&= n^2/12 - 1/3, & n \equiv 4(6)\\
&= n^2/12 - 1/12, & n \equiv 5(6).
\end{aligned}
$$

Let $Y = Y(k_1, k_2, k_3)$. *be a* $Y-$*tree where* $n = k_1 + k_2 + k_3 + 1$. *Since these three numbers characterize the* $Y-$*tree , the number of all* $Y-$*trees having order* n *is* $P_3(n-1)$.

In this section we will be interested in locating a vertex x_1 *which will minimize*

$\lambda_1(Y-v)$ for $v = x_1$, and moreover, we wish to characterize those $Y-$trees which have the property $\lambda_1(Y - x_1) = \lambda_2(Y)$. Clearly, the vertex x_1 which minimizes $\lambda_1(Y - v)$ is not necessarily unique (see Figure 11).

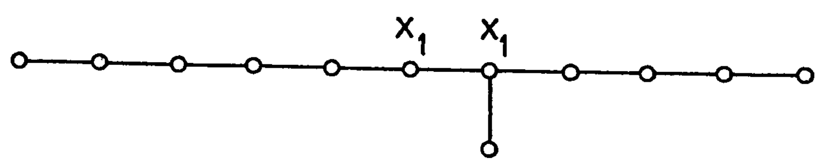


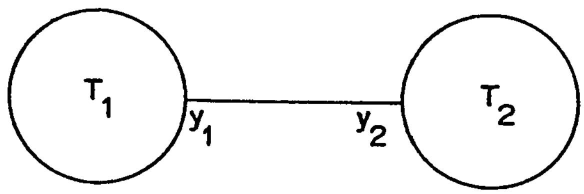
Figure 11: A tree $Y(6,4,1)$

Lemma 3.1.1 *If Y is a $Y-$tree, then $\lambda_1(Y - v) \geq \lambda_2(Y)$ for any vertex v in Y.*

Proof: *The proof is immediate by applying the interacting theorem.*

Theorem (2.1.3) can be rewritten for a $Y-$tree where $k = \lambda_2(Y)$.

Theorem 3.1.1 *Let Y be a $Y-$tree. Then either Y contains a vertex x such* *that* $\lambda_1(Y - x) = \lambda_2(Y)$; *or* Y *is a* λ*-twin ; i.e.,* Y *has the shape*



with subtrees T_1 *and* T_2 *satisfying*

$$\lambda_1(T_i - x_i) < \lambda_2(Y) < \lambda_1(T_i) \quad \textit{for} \quad i = 1, 2.$$

In theorem (3.1.1), the $Y-$ *tree which satisfies the first condition has this property* $\lambda_1(Y - x_1) = \lambda_2(Y)$*. By the above lemma and theorem, the only remaining possibility is for a* $Y-$ *tree to satisfy* $\lambda_1(Y - x_1) > \lambda_2(Y)$.

Remark 3.1.1 *Let* Y *be a* $Y-$ *tree and let* x *be a vertex in* Y*. Then*

$$Y - x = \begin{cases} \textit{union of three paths,} & \textit{if } x = u, \\ \textit{union of two paths ,} & \textit{if } x \sim u, x \neq \textit{ end vertex} \\ \textit{either one path or one Y-tree,} & \textit{if } x \textit{ is an end vertex} \\ \textit{union of a path and a Y-tree,} & \textit{otherwise} \end{cases}$$

where u *is the unique vertex of degree three, and* $u \sim x$ *means* $u \& x$ *are adjacent.*

3.2 A Characterization

The next lemmas and theorem will characterize those $Y-$ trees T which satisfy the property $\lambda_1(T-x_1)=\lambda_2(T)$. It turns out they are exactly the trees T satisfying

$$\lambda_1(T_1)=\lambda_1(T_2) \quad \textit{where} \quad T_1\cup T_2=T-x_1$$

Lemma 3.2.1 *Let x_1 be a vertex in a $Y-$ tree T satisfying $\lambda_1(T-x_1)=\lambda_2(T)$. Then x_1 cannot be an end-vertex of T.*

Proof: *Let $Y(k,\ell,m)$ be a $Y-$ tree where $k\geq\ell\geq m>0, n=k+\ell+m+1$.*

$$\begin{aligned}\lambda_1(Y-u) &= \max\{\lambda_1(P_k),\lambda_1(P_\ell),\lambda_1(P_m)\}\\ &= \lambda_1(P_k).\end{aligned}$$

Let v be an end vertex. Then

$$\lambda_1(Y-u)=\lambda_1(P_k)<\lambda_1(P_{n-1})<\lambda_1(Y-v).$$

The last inequality follows from theorem (2.1.2).

Lemma 3.2.2 *Let Y be a $Y-$ tree and let v be a vertex of degree less than three such that $Y-v=T_1\cup T_2$ and $\lambda_1(T_1)=\lambda_1(T_2)$ then:*

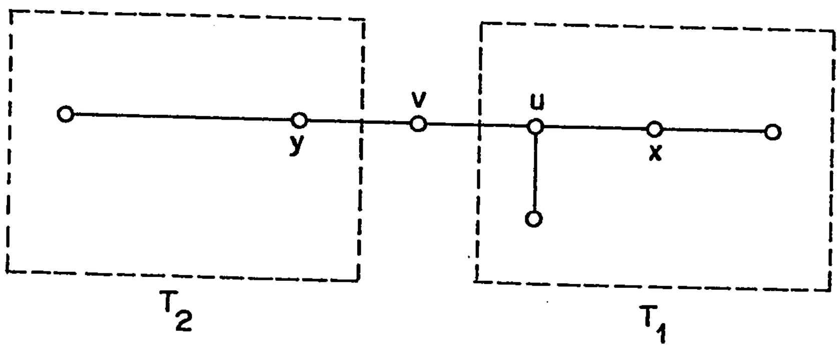
$$x_1=v \quad \textit{and} \quad \lambda_2(Y)=\lambda_1(T_1)=\lambda_1(T_2).$$

Proof: *Let Y be a $Y-$ tree and let v be a vertex such that $v\neq u$ and $\lambda_1(T_1)=\lambda_1(T_2)$ where $T_1\cup T_2=Y-v$. Let x be any vertex other than v. If*

$x \in V(T_1)$ then

$$\lambda_1(Y-x) = \max\{\lambda_1(T_1'), \lambda_1(T_2')\}$$

where $T_1' \cup T_2' = Y - x$.



Since T_1' is a subgraph of T_1 and T_2 is a subgraph of T_2' then

$$\lambda_1(T_2') > \lambda_1(T_2) = \lambda_1(T_1) > \lambda_1(T_1').$$

Hence,

$$\lambda_1(Y-x) = \max\{\lambda_1(T_1'), \lambda_1(T_2')\} = \lambda_1(T_2')$$

Since $\lambda_1(Y-x) = \lambda_1(T_2') > \lambda_1(T_1) = \lambda_1(Y-v)$, then $x_1 \neq x$. If $x \in V(T_2)$, then

$$\lambda_1\ (Y-x) = \max\{\lambda_1(T_1''), \lambda_1\lambda_1(T_2'')\} \qquad \text{where } T_1'' \cup T_2'' = Y - x.$$

Since T_1 is a subgraph of T_1'' and T_2'' is a subgraph of T_2, it follows that

$$\lambda_1(T_1'') > \lambda_1(T_1) = \lambda_1(T_2) > \lambda_1(T_2'')$$

Hence,

$$\lambda_1(Y-x) = \max\{\lambda_1(T_1''), \lambda_1(T_2'')\} = \lambda_1(T_1'')$$

Since $\lambda_1(Y-x) = \lambda_1(T_1'') > \lambda_1(T_1) = \lambda_1(Y-v)$, then $x_1 \neq x$. Therefore, $x_1 = v$. Suppose that x and y are adjacent to v and $x \in V(T_2)$ and $y \in V(T_1)$. Then

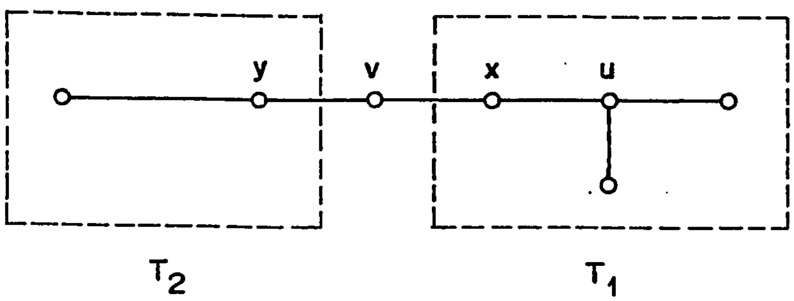
$$\begin{aligned}\phi(Y;\lambda) &= \lambda \cdot \phi(T_1;\lambda) \cdot \phi(T_2;\lambda) - \phi(T_1;\lambda) \cdot \phi(T_2 - x;\lambda) \\ &- \phi(T_2;\lambda) \cdot \phi(T_1 - y;\lambda).\end{aligned}$$

Let $d = \lambda_1(T_1) = \lambda_1(T_2)$ *then* $\phi(T_1; d) = \phi(T_2; d) = 0$. *which implies that* $\phi(Y; d) = 0$. *Hence,* d *is an eigenvalue of* Y. *Since* $x_1 = v$ *then* $\lambda_1(Y - v) = \lambda_2(Y)$ *using theorem (1.2.11) we get* $\lambda_1(Y - v) < \lambda_1(Y)$. *Hence,* $\lambda_1(Y - v) = \lambda_2(Y)$.

Lemma 3.2.3 *Let* Y *be a* $Y-$*tree and let* v *be a vertex such that* $\lambda_1(Y - v) = \lambda_2(Y)$. *Then* $\lambda_1(T_1) = \lambda_1(T_2)$ *where* $T_1 \cup T_2 = Y - v$.

Proof: *Let* Y *be a* $Y-$*tree and let* v *be a vertex such that* $\lambda_1(Y - v) = \lambda_2(Y) = d$. *Suppose that* $\lambda_1(T_1) \neq \lambda_1(T_2)$ *without loss of generality* $\lambda_1(T_1) < \lambda_1(T_2)$. *So,*

$$\begin{aligned} \lambda_1(Y - v) &= \max\{\lambda_1(T_1), \lambda_1(T_2)\} \\ &= \lambda_1(T_2) \end{aligned}$$



$$\begin{aligned} \phi(Y - v; \lambda) &= \phi(T_1; \lambda) \cdot \phi(T_2; \lambda) \\ \phi(Y; \lambda) &= \lambda \cdot (T_1; \lambda) \cdot \phi(T_2; \lambda) - \phi(T_1 - x; \lambda) \cdot \phi(T_2; \lambda) \\ &- \phi(T_2 - y; \lambda) \cdot \phi(T_1; \lambda) \end{aligned}$$

where $x \sim v, y \sim v$.

$$\phi(Y; d) = d\phi(T_1; d) \cdot \phi(T_2; d) - \phi(T_1 - x; d) \cdot \phi(T_2; d)$$

$$- \quad \phi(T_2 - y; d) \cdot \phi(T_1; d)$$

$$= \quad d\phi(T_1; d) \cdot [0] - \phi(T_1 - x; d) \cdot [0]$$

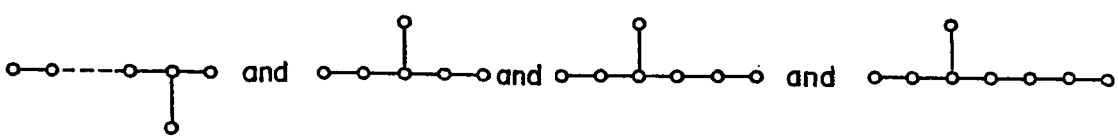
$$- \quad \phi(T_2 - y; d) \cdot \phi(T_1; d) = 0$$

Hence,

$$\phi(T_2 - y; d) \cdot \phi(T_1; d) = 0 \tag{3.1}$$

Since $\lambda_1(T_2 - y) < \lambda_1(T_2) = \lambda_2(Y) = d$ *it follows that* d *is not an eigenvalue of* $T_2 - y$ *i.e.,* $\phi(T_2 - y; d) \neq 0$. *From (3.1),* $\phi(T_1; d) = 0$ *implies that* d *is an eigenvalue of* T_1, *but* $d = \lambda_1(T_2) > \lambda_1(T_1)$. *(Contradiction).*

Remark 3.2.1 *From theorem (1.2.14), all* $Y-$*trees with* $\lambda_1 < 2$ *are:*



Y(k,1,1) Y(2,2,1) Y(3,2,1) Y(4,2,1)

Theorem 3.2.1 *If* Y *is a* $Y-$*tree with* $\lambda_1(T - x_1) = \lambda_2(T)$ *where* $x_1 \neq u$, *then* T *is one of the trees in Figure 13.*

Proof: *Let* Y *be a* $Y-$ *tree with* $\lambda_1(Y - x_1) = \lambda_2(T)$ *where* $x_1 \neq u$, *then* $Y - x_1 = T_1 \cup T_2$ *There are two cases:*

(i) T_1 *and* T_2 *are paths. The remark after theorem (3.1.1) implies that* x_1 *is adjacent to* u*. By lemma (3.2.3) ,* $\lambda_1(T_1) = \lambda_1(T_2)$ *which implies* $T_1 = T_2$*. Hence, the shape of* Y *is as in Figure 13(A).*

(ii) Only one of T_1 *or* T_2 *is a path. Without loss of generality, let* T_1 *be a path and* T_2 *a* $Y-$*tree. By lemma (3.2.3), we have* $\lambda_1(T_2) = \lambda_1(T_1)$*. Since* T_1 *is a path then* $\lambda_1(T_1) < 2$ *implies* $\lambda_1(T_2) < 2$*. Hence,* T_2 *is one of the graphs in the above remark. In what follows we will find for each of the four* $Y-$*trees* T *in the above remark the corresponding path* P *such that* $\lambda_1(T) = \lambda_1(P)$*.*

Consider the first $Y-$*tree* $Y(k-1,1,1)$ *in the above remark, using theorem (3.1.1) we have*

$$\lambda_1[Y(k-1,1,1)] = 2\cos\frac{\pi}{2(k+2)-2} = 2\cos\frac{\pi}{(2k+1)+1} = \lambda_1[P_{2k+1}]$$

Therefore, the corresponding path of $Y(k-1,1,1)$ *is* P_{2k+1}*.*

To construct the tree Y *which satisfies*

$$Y - x_1 = Y(k-1,1,1) \cup P_{2k+1},$$

the vertex x_1 *is simultaneously joined to one end vertex of* $Y(k-1,1,1)$ *and an end vertex of* P_{2k+1} *as shown in Figure 12. Then we get two non-isomorphic trees* $Y(3k+1,1,1)$ *and* $Y(2k+3,k-1,1)$*.*

Similarly, the following trees and their corresponding paths are obtained:

Figure 3(B,C) $\quad \lambda_1[Y(k-1,1,1)] = 2\cos\dfrac{\pi}{(2k+1)+1} = \lambda_1[P_{2k+1}]$

Figure 3(D) $\quad \lambda_1[Y(4,2,1)] = 2\cos\dfrac{\pi}{30} = \lambda_1[P_{29}]$

Figure 3(E) $\lambda_1[Y(3,2,1)] = 2\cos\frac{\pi}{18} = \lambda_1[P_{17}]$

Figure 3(F) $\lambda_1[Y(2,2,1)] = 2\cos\frac{\pi}{12} = \lambda_1[P_{11}]$

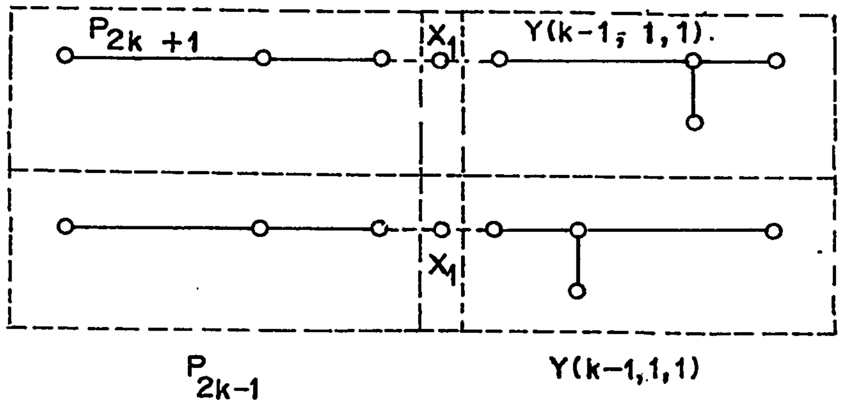


Figure 12

Theorem 3.2.2 *Let $Y(k,\ell,m)$ be a $Y-$tree. where $k \geq \ell \geq m > 0$ such that $\lambda_1(Y-u) = \lambda_2(Y)$, i.e., $x_1 = u$. Thus $\ell = k$ or $m = \ell = k$.*

Proof : *Let $Y(k,\ell,m)$ be a $Y-$tree where $k \geq \ell \geq m > 0$ such that $\lambda_1(Y-u) = \lambda_2(Y) = d$, i.e., $x_1 = u$.*

$$\lambda_1(Y-u) = \max\{\lambda_1(P_k), \lambda_1(P_\ell), \lambda_1(P_m)\} = \lambda_1(P_k) = d$$

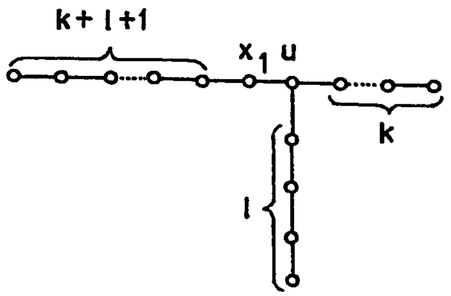
$$\begin{aligned}\phi(Y-u,\lambda) &= \lambda\phi(P_k;\lambda)\cdot\phi(P_\ell;\lambda)\cdot\phi(P_m;\lambda)\\ &- \phi(P_{k-1};\lambda)\cdot\phi(P_\ell;\lambda)\cdot\phi(P_m;\lambda)\\ &- \phi(P_k;\lambda)\cdot\phi(P_{\ell-1};\lambda)\cdot\phi(P_m;\lambda)\\ &- \phi(P_k;\lambda)\cdot\phi(P_\ell;\lambda)\cdot\phi(P_{m-1};\lambda)\end{aligned}$$

Since $\phi(P_k;d)=0$, *then*

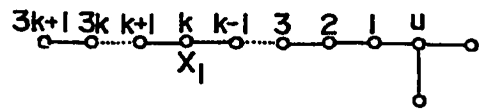
$$\phi(Y-u;d) = -\phi(P_{k-1};d)\cdot\phi(P_\ell;d)\cdot\phi(P_m;d) = 0$$

But $\phi(P_{k-1};d)\neq 0$ *because* $d=\lambda_1(P_k)>\lambda_1(P_{k-1})$. *Then* $\phi(P_\ell;d)=0$ *or* $\phi(P_m;d)=0$ *which implies that* $d=\lambda_1(P_\ell)$ *or* $d=\lambda_1(P_m)$. *Therefore,* $\ell=k$ *or* $m=k$.

Remark 3.2.2 *From theorem (3.2.2), if* $x_1=u$ *and* $\lambda_1(Y-u)=\lambda_2(Y)$ *then* Y *is one of the trees in Figure 14.*

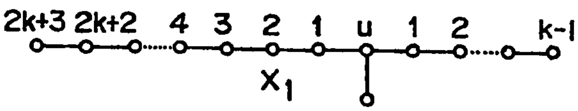


A. $n=2k+2\ell+3$



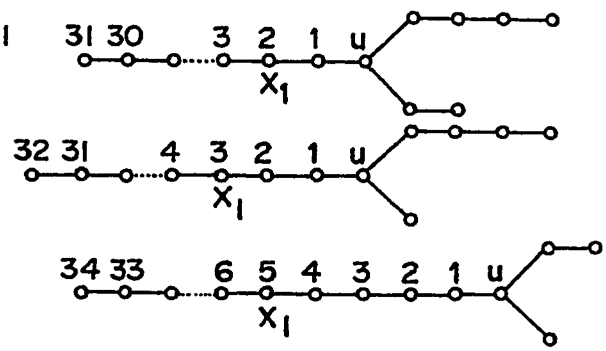
B. $\lambda_1[P_{2k+1}]=\lambda_1[P^*_{k+2}]$

where $k=2,3,4,\ldots.$
$n=3k+1$

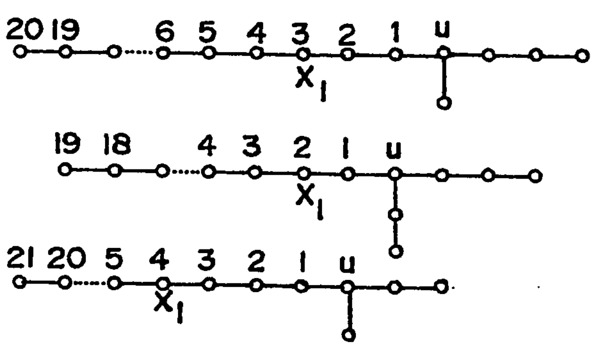


C. $\lambda_1[P_{2k+1}]=\lambda_1[P^*_{k+2}]$

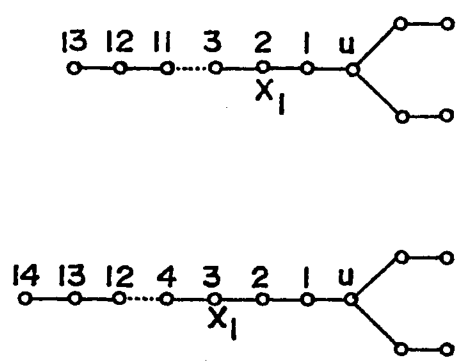
where $k=2,3,4,\ldots$
$n=3k+4$



D. $\lambda_1[P_{29}]=\lambda_1[P^*(4,2,1)]$

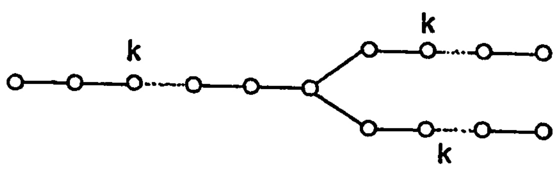


E. $\lambda_1[P_{17}]=\lambda_1[P^*(3,2,1)]$



F. $\lambda_1[P_{11}]=\lambda_1[P^*(2,2,1)]$

FIGURE 13



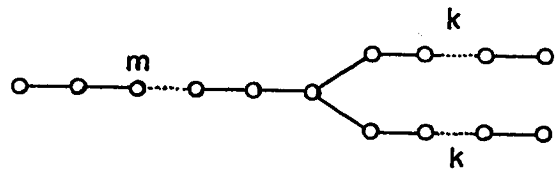


Figure 14

Remark 3.2.3 *From theorem (3.2.1) and (3.2.2), all $Y-$trees with $\lambda_1(Y-x_1) = \lambda_2(Y)$ are in Figure 13 and Figure 14.*

3.3 Bounds On The Second Largest Eigenvalue

Let T be a member of the family of all trees with three end vertices having a unique vertex of degree 3. Here we will attempt to present an upper and a lower bound on the second largest eigenvalue in the spectrum of T. Also, we draw all trees which minimize λ_2 for large enough n.

Theorem 3.3.1 *Let $Y(k_1, k_2, k_3)$ be a $Y-$tree of order $n \geq 4$ where $n = k_1 + k_2 + k_3 + 1, k_1 \geq k_2 \geq k_3 > 0$ then:*

$$\lambda_2(P_{k_1}) \leq \lambda_2(Y) \leq \lambda_1(P_{k_1})$$

Proof: *Let $Y(k_1, k_2, k_3)$ be a $Y -$ tree of order n with the shape as shown in Figure 15, and let F be a forest of order $n-1$ obtained from Y by deleting the unique vertex u as in Figure 16.*

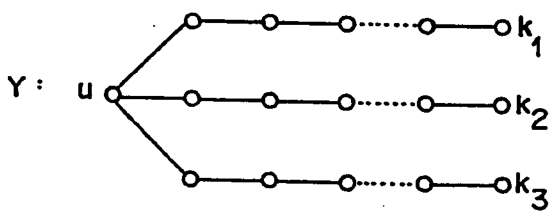


Figure 15 Y-tree of order n

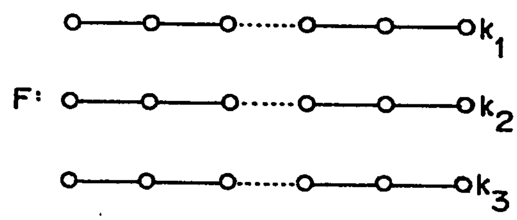


Figure 16. A forest of order $n-1$

Since $F = Y - u$ is a subgraph of $Y(k_1, k_2, k_3)$, then by the interlacing theorem we have

$$\lambda_2(F) \leq \lambda_2(Y) \leq \lambda_1(F) = \lambda_1(P_{k_1}) \tag{3.2}$$

By the property of eigenvalues of unions of disjoint graph we obtain

$$\lambda_2(F) = \max\{\lambda_1(P_{k_2}), \lambda_1(P_{k_3}), \lambda_2(P_{k_1})\} \geq \lambda_2(P_{k_1})$$

which implies that

$$\lambda_2(F) \geq \lambda_2(P_{k_1})$$

Hence,

$$\lambda_2(Y) \geq \lambda_2(F) \geq \lambda_2(P_{k_1}) \tag{3.3}$$

and from (3.2) and (3.3), we get

$$\lambda_2(P_{k_1}) \leq \lambda_2(Y) \leq \lambda_1(P_{k_1})$$

Theorem 3.3.2 *Let $Y(k_1, k_2, k_3)$ be a $Y-tree$ of order $n \geq 4$ where $n = k_1 + k_2 + k_3 + 1, k_1 \geq k_2 \geq k_3 > 0$ then*

$$\lambda_2(Y) \leq \lambda_1(P_{n-3}) = 2\cos(\frac{\pi}{n-2})$$

Proof: *From theorem (3.3.1)*

$$\lambda_2(Y) \leq \lambda_1(P_{k_1}) \textit{ where } Y = Y(k_1, k_2, k_3)$$

Since the maximum value for k_1 is $n-3$ for all $Y-tree$ with order n, then

$$\lambda_2(Y) \leq \lambda_1(P_{k_1}) \leq \lambda_1(P_{n-3}) = 2\cos(\frac{\pi}{n-2})$$

Theorem 3.3.3 *Let $Y(k_1, k_2, k_3)$ be a $Y-tree$ where $n = k_1 + k_2 + k_3 + 1, k_1 \geq k_2 \geq k_3 > 0$, then*

$$\lambda_2(Y) \geq \lambda$$

where

$$\lambda = \begin{cases} 2\cos(\frac{3\pi}{n+3}) \textit{ if } n = 3k \\ 2\cos(\frac{3\pi}{n+2}) \textit{ if } n = 3k+1 \\ 2\cos(\frac{6\pi}{2n+5}) \textit{ if } n = 3k+2 \end{cases}$$

Proof: *Let Y be a $Y-tree$ of order n with the shape as shown in Figure 17, and let F be the forest $Y - x$ of order $n-1$ where x is adjacent to the unique* *vertex u as shown in Figure 18.*

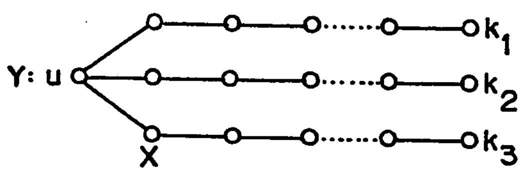


Figure 17 A $Y-tree$ of order n

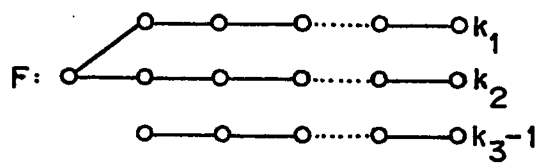


Figure 18 A Forest of order $n-1$

Since $F = Y - x$ is a subgraph of $Y(k_1, k_2, k_3)$ then by the interlacing theorem, we have

$$\begin{aligned}\lambda_2(Y) \geq \lambda_2(F) &= \max\{\lambda_2(P_{k_1+k_2+1}), \lambda_1(P_{k_3-1})\} \\ &= \lambda_2(P_{k_1+k_2+1})\end{aligned}$$

By considering the three cases

(i) $\underline{n = 3k}$

In this case, the minimum value for $k_1 + k_2$ is $\frac{n}{3} + \frac{n}{3}$.

Therefore, we get

$$\begin{aligned}\lambda_2(Y) &\geq \lambda_2(P_{k_1+k_2+1}) \geq \lambda_2(P_{\frac{n}{3}+\frac{n}{3}+1}) \\ &= 2\cos(\frac{2\pi}{\frac{n}{3}+\frac{n}{3}+2}) = 2\cos(\frac{3\pi}{n+3})\end{aligned}$$

(ii) $\underline{n = 3k + 1}$

In this case, the minimum value for $k_1 + k_2$ *is* $\frac{2(n-1)}{3}$. *We have*

$$\begin{aligned}
\lambda_2(Y) \geq \lambda_2(P_{k_1+k_2+1}) \geq \lambda_2(P_{\frac{2(n-1)}{3}+1}) &= 2\cos(\frac{2\pi}{\frac{2(n-1)}{3}+2}) \\
&= 2\cos(\frac{3\pi}{n+2}).
\end{aligned}$$

(iii) $\underline{n = 3k + 2}$

In this case, the minimum value for $k_1 + k_2$ *is* $\frac{n+1}{3} + \frac{n-2}{3}$. *It follows that*

$$\begin{aligned}
\lambda_2(Y) \geq \lambda_2(P_{k_1+k_2+1}) &\geq \lambda_2(P_{\frac{n+1}{2}+\frac{n-2}{3}+1}) \\
&= 2\cos(\frac{2\pi}{\frac{n+1}{3}+\frac{n-2}{3}+2}) \\
&= 2\cos(\frac{6\pi}{2n+5})
\end{aligned}$$

Remark 3.3.1 *(1) In case* $n = 3k$, *the minimum is attained when* Y *has the structure shown in Figure 19.*

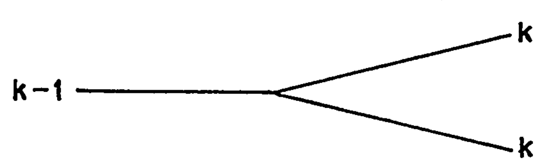


Figure 19 λ_2 **is minimum when** $n = 3k$

(2) In case $n = 3k+1$, the minimum is attained when Y has the structure shown in Figure 20.

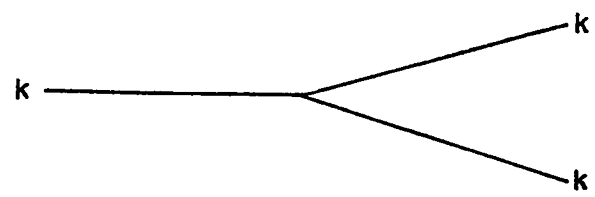


Figure 20 λ_2 **is minimum when** $n = 3k+1$

(3) There is no $Y-$tree of the third type that attains the minimum. However, the tree $Y(k+1,k,k)$ in Figure 21 has the smallest possible value for λ_2 but does not attain $2\cos(\frac{6\pi}{2n+5})$.

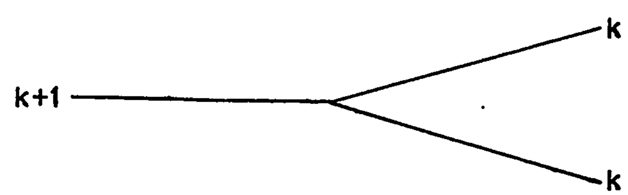


Figure 21 A $Y - treeY(k+1,k,k)$

(4) Among the family of all trees T, the maximum of λ_1 is assumed when T is a star and the minimum is attained when T is a path [12]. However, for the family of $Y-$trees, we conjuctured that the maximum and minimum

of λ_1 *are attained for the trees shown in Figure (19,20,21) and Figure (22) respectively.*

For the case of λ_2 *we have shown that the* $Y-tree$ *of Figure (19,20,21) gives a minimum whereas the maximum case is still an open problem.*

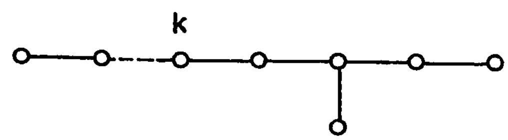
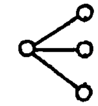
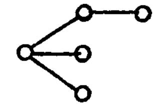
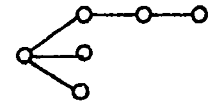
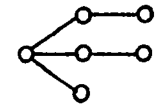
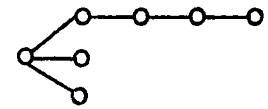
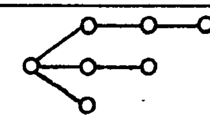
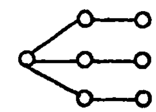
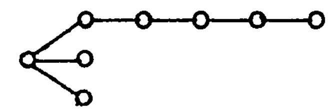
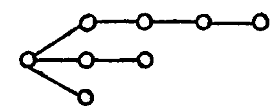
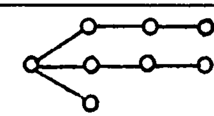


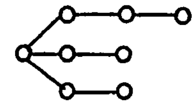
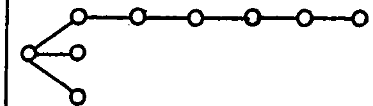
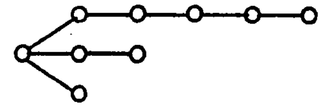
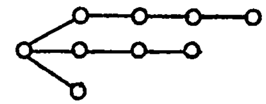
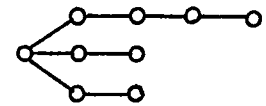
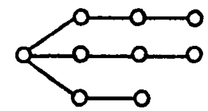
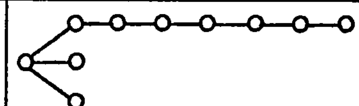
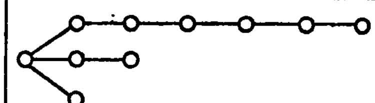
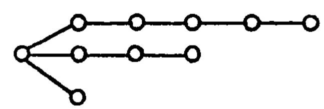
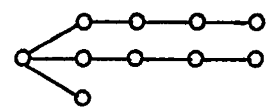
Figure 22 **A** $Y - treeY(k, 2, 1\ )$

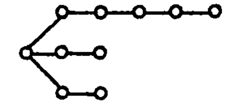
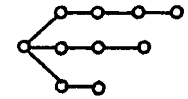
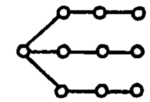
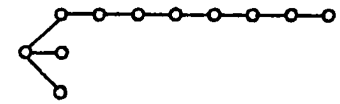
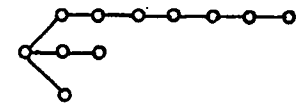
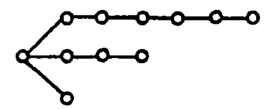
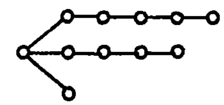
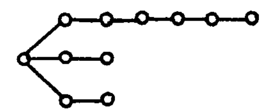
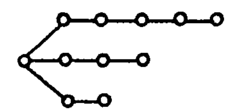
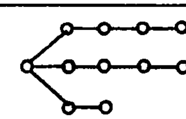
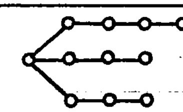
APPENDIX

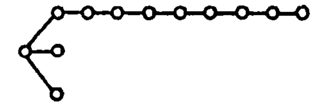
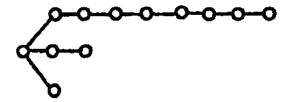
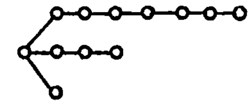
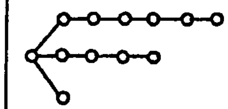
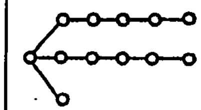
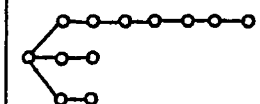
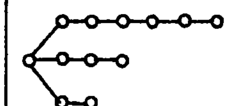
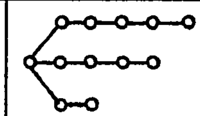
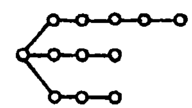
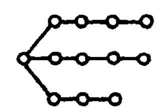
All Y-trees of order $n \leq 18$ and their non-negative eigenvalues.

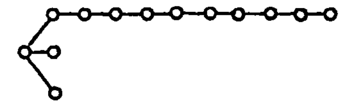
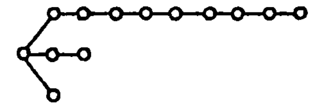
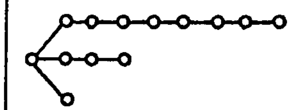
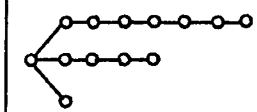
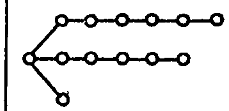
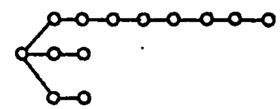
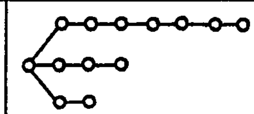
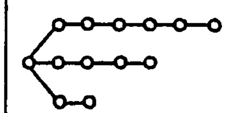
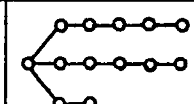
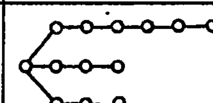
$$[\lambda_i = \lambda_{n+1-i} \quad \text{for} \quad 1 \leq i \leq \frac{1}{2}(n-1)].$$

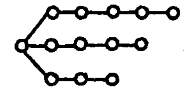
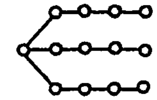
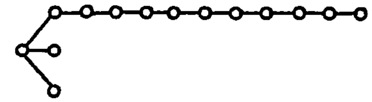
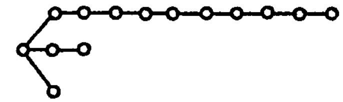
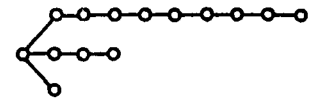
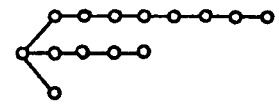
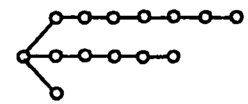
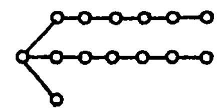
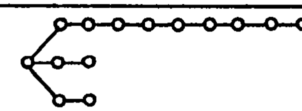
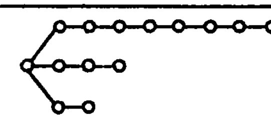
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(1,1,1)	1.73205	0	0						
Y(2,1,1)	1.847759	.7653668	0						
Y(3,1,1)	1.90211	1.17557	0	0					
Y(2,2,1)	1.93185	1	.517638						
Y(4,1,1)	1.93185	1.41421	.517638	0					
Y(3,2,1)	1.96962	1.28558	.68404	0					
Y(2,2,2)	2	1	1	0					
Y(5,1,1)	1.94986	1.56366	.867767	0	0				
Y(4,2,1)	1.98904	1.48629	.813473	.415823					
Y(3,3,1)	2	1.41421	1	0	0				

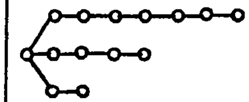
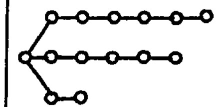
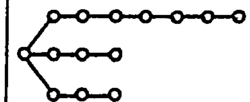
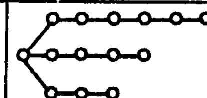
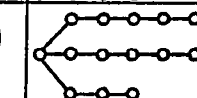
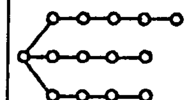
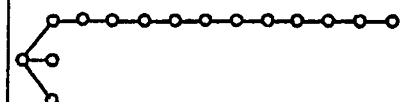
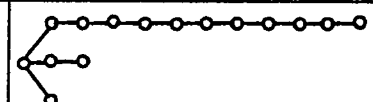
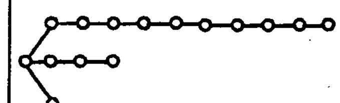
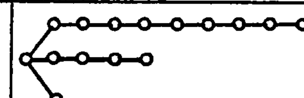
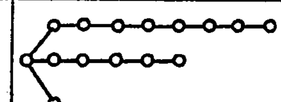
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(3,2,2)	2.02852	1.32132	1	.373087					
Y(6,1,1)	1.96157	1.66294	1.11114	.390181	0				
Y(5,2,1)	2	1.61803	1	.618034	0				
Y(4,3,1)	2.01532	1.54801	1.14288	.485783	0				
Y(4,2,2)	2.042081	1.52023	1	.720282	0				
Y(3,3,2)	2.05288	1.41421	1.20864	.569973	0				
Y(7,1,1)	1.96961	1.73205	1.28557	.68404	0	0			
Y(6,2,1)	2.00659	1.70697	1.18968	.728783	.336735				
Y(5,3,1)	2.02368	1.65458	1.23399	.802699	0	0			
Y(4,4,1)	2.02852	1.61803	1.32132	.618034	.373087				

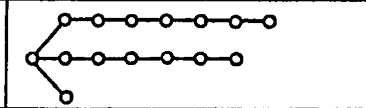
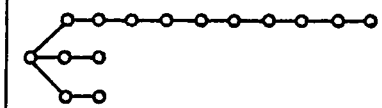
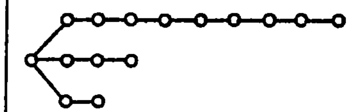
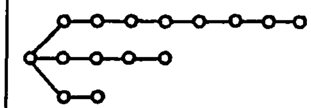
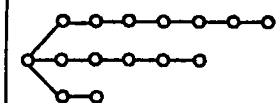
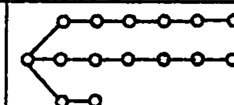
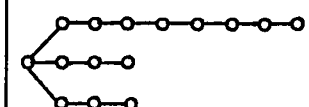
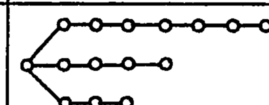
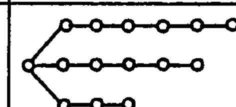
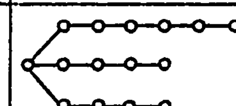
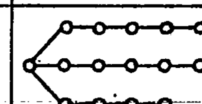
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(5,2,2)	2.04907	1.64728	1	1	.296261				
Y(4,3,2)	2.06416	1.55744	1.26788	.779055	.314919				
Y(3,3,3)	2.07431	1.41421	1.41421	.835	0	0			
Y(8,1,1)	1.97536	1.78201	1.41421	.907994	.312868	0			
Y(7,2,1)	2.01074	1.76922	1.34167	.850388	.550914	0			
Y(6,3,1)	2.02852	1.73205	1.32132	1	.373087	0			
Y(5,4,1)	2.03565	1.69069	1.41421	.884133	.464762	0			
Y(6,2,2)	2.05288	1.73205	1.20864	1	.569973	0			
Y(5,3,2)	2.06978	1.66891	1.29674	1	.499199	0			
Y(4,4,2)	2.07431	1.61803	1.41421	.834999	.618034	0			
Y(4,3,3)	2.08397	1.57184	1.41421	1	.431733	0			

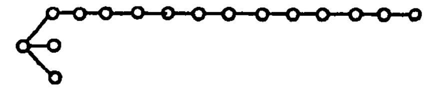
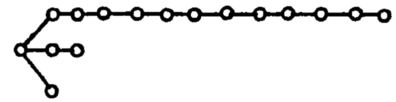
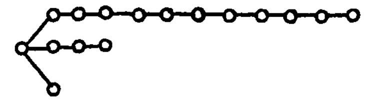
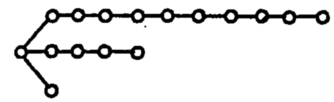
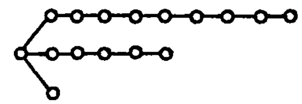
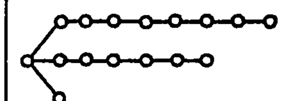
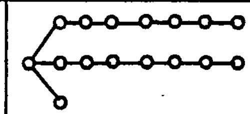
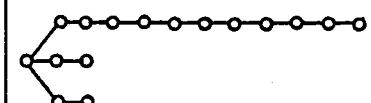
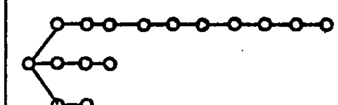
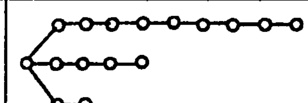
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(9,1,1)	1.97964	1.81926	1.5115	1.08128	.563466	0	0		
Y(8,2,1)	2.01348	1.81425	1.45826	1	.670428	.280007			
Y(7,3,1)	2.03144	1.78805	1.41421	1.11917	.650834	0	0		
Y(6,4,1)	2.03969	1.75313	1.47219	1.0979	.53884	.321095			
Y(5,5,1)	2.04208	1.73205	1.52023	1	.720282	0	0		
Y(7,2,2)	2.05504	1.7909	1.36293	1	.806771	.247108			
Y(6,3,2)	2.07272	1.74687	1.33633	1.1658	.65965	.268747			
Y(5,4,2)	2.0793	1.69458	1.47563	1	.749346	.256661			
Y(5,3,3)	2.08862	1.68101	1.41421	1.14907	.701084	0	0		
Y(4,4,3)	2.09282	1.61803	1.51373	1.11758	.618034	.282452			

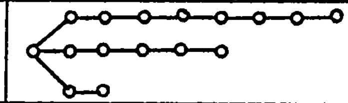
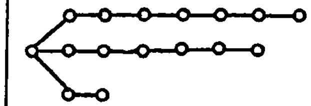
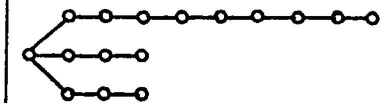
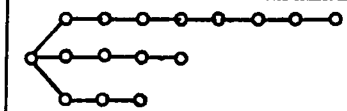
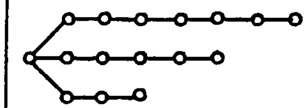
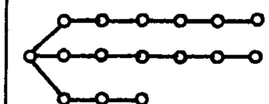
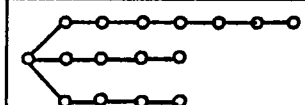
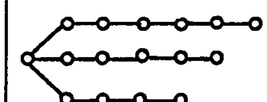
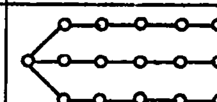
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(10,1,1)	1.98289	1.84776	1.58671	1.21752	.765367	.261052	0		
Y(9,2,1)	2.0153	1.84776	1.54801	1.14288	.765367	.485783	0		
Y(8,3,1)	2.03326	1.8292	1.50236	1.20054	.854516	.302159	0		
Y(7,4,1)	2.04208	1.80194	1.52023	1.24698	.720281	.445042	0		
Y(6,5,1)	2.0457	1.7757	1.58023	1.16271	.825324	.363081	0		
Y(8,2,2)	2.05628	1.83314	1.47847	1	1	.47474	0		
Y(7,3,2)	2.07431	1.80194	1.41421	1.24698	.834997	.445041	0		
Y(6,4,2)	2.08187	1.76056	1.50805	1.18553	.791334	.510218	0		
Y(5,5,2)	2.08397	1.73205	1.57184	1	1	.431733	0		
Y(6,3,3)	2.09096	1.75631	1.41421	1.29129	.87095	.342412	0		

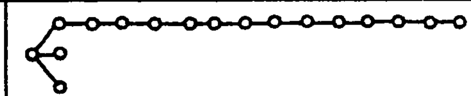
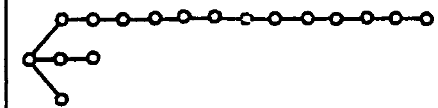
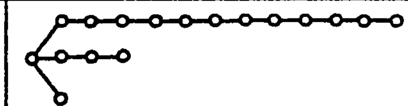
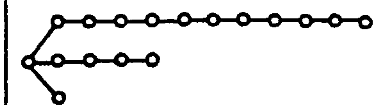
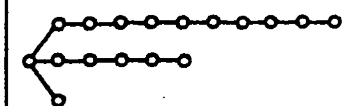
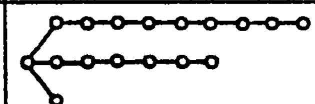
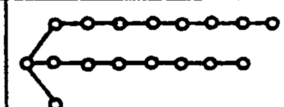
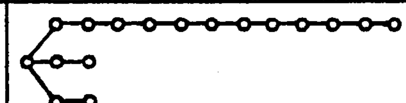
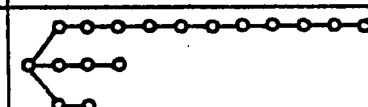
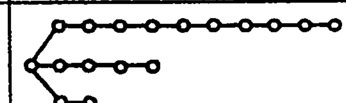
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(5,4,3)	2.09702	1.69843	1.54135	1.21264	.844636	.397676	0		
Y(4,4,4)	2.101	1.61803	1.61803	1.25928	.618034	.618034	0		
Y(11,1,1)	1.98542	1.87003	1.64597	1.32625	.929446	.478632	0	0	
Y(10,2,1)	2.01656	1.8733	1.61803	1.26401	.877175	.618034	.238749		
Y(9,3,1)	2.03441	1.86008	1.57791	1.27142	1	.543103	0	0	
Y(8,4,1)	2.04352	1.83918	1.56843	1.3472	.916212	.502178	.273682		
Y(7,5,1)	2.04781	1.8152	1.61803	1.30577	.898038	.618034	0	0	
Y(6,6,1)	2.04907	1.80194	1.64728	1.24698	1	.445042	.296261		
Y(9,2,2)	2.05703	1.86432	1.56655	1.15478	1	.678313	.212501		
Y(8,3,2)	2.07519	1.84177	1.50582	1.28042	1	.583415	.232596		

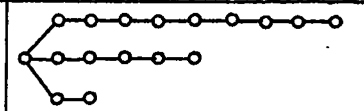
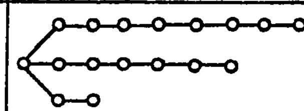
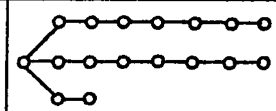
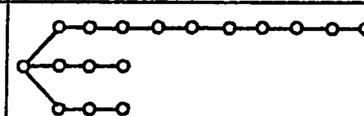
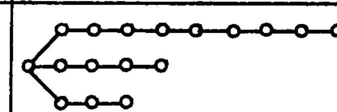
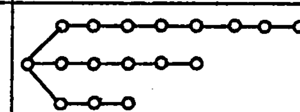
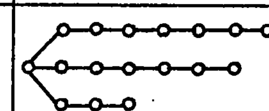
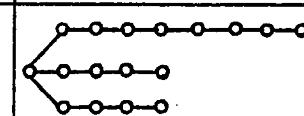
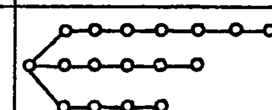
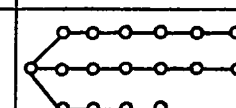
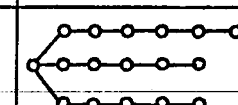
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(7,4,2)	2.08323	1.81055	1.5363	1.32338	.858858	0.696685	0.217939		
Y(6,5,2)	2.08635	1.77772	1.61803	1.19844	1	0.618034	0.224974		
Y(7,3,3)	2.09217	1.80939	1.41421	1.41421	1	.590676	0	0	
Y(6,4,3)	2.09911	1.7658	1.55181	1.31416	1	.526748	.251149		
Y(5,5,3)	2.101	1.73205	1.61803	1.25928	1	.68034	0	0	
Y(5,4,4)	2.10484	1.70571	1.61803	1.344	.894455	.618034	.231702		
Y(12,1,1)	1.98741	1.88777	1.69345	1.41421	1.06406	.660558	.22936	0	
Y(11,2,1)	2.01742	1.89317	1.67349	1.36384	1	.70812	.42861	0	
Y(10,3,1)	2.03515	1.8837	1.64006	1.34296	1.10145	.734085	.253694	0	
Y(9,4,1)	2.04442	1.86767	1.61803	1.41421	1.07417	.618034	.422299	0	
Y(8,5,1)	2.04907	1.84776	1.64728	1.41421	1	.765367	.296261	0	

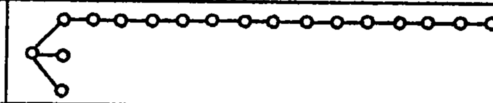
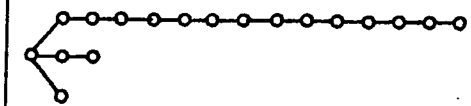
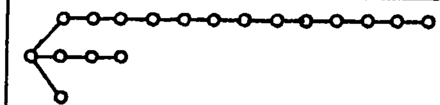
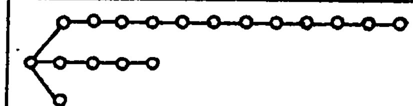
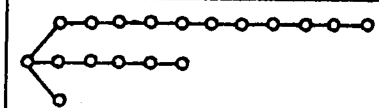
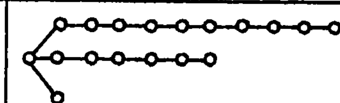
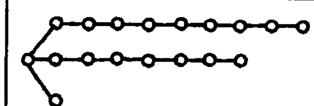
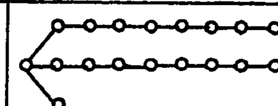
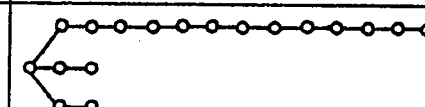
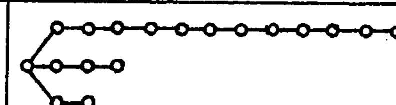
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	Graph
Y(7,6,1)	2.05102	1.83014	1.68735	1.35371	1.08706	.675567	.355123	0	
Y(10,2,2)	2.05748	1.88793	1.6349	1.27878	1	.852997	.408309	0	
Y(9,3,2)	2.07568	1.87129	1.58451	1.30739	1.13289	.725521	.400044	0	
Y(8,4,2)	2.08397	1.84776	1.57184	1.41421	1	.765367	.431733	0	
Y(7,5,2)	2.08759	1.81959	1.6431	1.34616	1	.81941	.384297	0	
Y(6,6,2)	2.08862	1.80194	1.68101	1.24698	1.14907	.701084	.445042	0	
Y(8,3,3)	2.09282	1.84776	1.51373	1.41421	1.11758	.765367	.282452	0	
Y(7,4,3)	2.10017	1.8155	1.56121	1.41421	1.09984	.713165	.370961	0	
Y(6,5,3)	2.10296	1.7793	1.64978	1.32447	1.12537	.751387	.323428	0	
Y(6,4,4)	2.10672	1.7716	1.61803	1.41421	1.08616	.618034	.493359	0	
Y(5,5,4)	2.1085	1.73205	1.67237	1.41421	1	.793519	.357385	0	

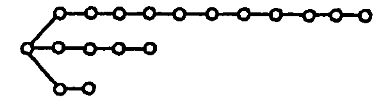
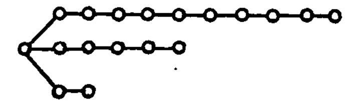
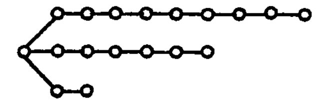
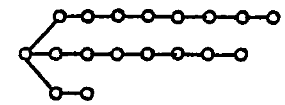
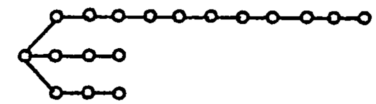
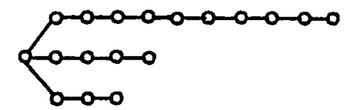
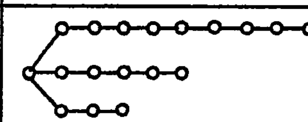
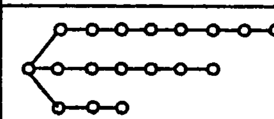
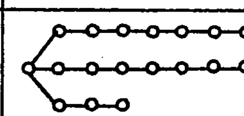
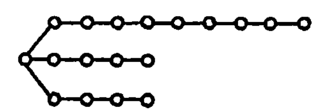
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8		Graph
Y(13,1,1)	1.98901	1.90209	1.73204	1.48629	1.17557	.813473	.415844	0	0	
Y(12,2,1)	2.01807	1.90887	1.71804	1.44595	1.1149	.796034	.566699	.207791		
Y(11,3,1)	2.03565	1.90203	1.69069	1.41421	1.17557	.884132	.464763	0	0	
Y(10,4,1)	2.04497	1.88971	1.66557	1.46265	1.19684	.784375	.479875	.235787		
Y(9,5,1)	2.04984	1.87367	1.67462	1.4931	1.12561	.84128	.526807	0	0	
Y(8,6,1)	2.05218	1.85665	1.71296	1.45254	1.14486	.869163	.402042	.263666		
Y(7,7,1)	2.05288	1.84776	1.73205	1.41421	1.20864	.765367	.569973	0	0	
Y(11,2,2)	2.05773	1.90605	1.68884	1.37884	1	1	.586603	.186644		
Y(10,3,2)	2.07596	1.89363	1.64806	1.3496	1.22106	.869415	.527613	.204185		
Y(9,4,2)	2.08437	1.87572	1.61803	1.46683	1.14137	.804166	.618034	.189979		

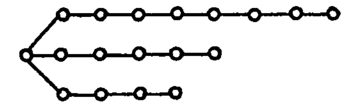
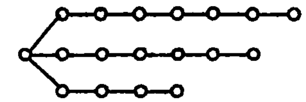
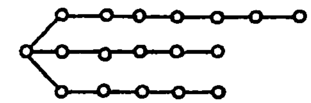
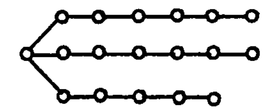
Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8		Graph
Y(8,5,2)	2.08826	1.85324	1.66189	1.45446	1	1	.536294	.199331		
Y(7,6,2)	2.08981	1.83133	1.7138	1.36749	1.18112	.838887	.579002	.194342		
Y(9,3,3)	2.09316	1.87617	1.59266	1.41421	1.22837	.893683	.504523	0	0	
Y(8,4,3)	2.10072	1.85207	1.57851	1.49262	1.17988	.879071	.471865	.222893		
Y(7,5,3)	2.10394	1.82234	1.66462	1.41421	1.19362	.873729	.541693	0	0	
Y(6,6,3)	2.10485	1.80194	1.70577	1.344	1.24698	.894396	.445038	.231702		
Y(7,4,4)	2.10766	1.81978	1.61803	1.48463	1.21141	.731614	.618034	.198145		
Y(6,5,4)	2.11027	1.78129	1.68743	1.46424	1.15605	.855094	.520477	.209265		
Y(5,5,5)	2.11199	1.73205	1.73205	1.49637	1	1	.548062	0	0	

Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8		Graph
Y(14,1,1)	1.99022	1.91385	1.76392	1.54602	1.26879	.942796	.58057	.196057	0	
Y(13,2,1)	2.0185	1.92153	1.7543	1.51386	1.21546	.896613	.661374	.380964	0	
Y(12,3,1)	2.03589	1.91669	1.73205	1.48045	1.23784	1	.639811	.218577	0	
Y(11,4,1)	2.04532	1.90699	1.70807	1.50317	1.29063	.934402	.555386	.394444	0	
Y(10,5,1)	2.05031	1.89416	1.70289	1.549	1.24079	.909807	.692383	.249784	0	
Y(9,6,1)	2.05288	1.87938	1.73205	1.53209	1.20864	1	.569973	.347297	0	
Y(8,7,1)	2.05398	1.86701	1.75981	1.4879	1.27578	.923461	.647324	.292004	0	
Y(12,2,2)	2.05784	1.9204	1.73205	1.46035	1.12309	1	.744342	.358996	0	
Y(11,3,2)	2.07606	1.91085	1.69911	1.41421	1.26476	1	.648682	.361646	0	
Y(10,4,2)	2.8459	1.89704	1.66685	1.49829	1.2571	.880586	.732271	.374727	0	

Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8		Graph
Y(9,5,2)	2.08862	1.87938	1.68099	1.53209	1.14907	1	.701084	.347292	0	
Y(8,6,2)	2.09044	1.85946	1.73205	1.4739	1.19285	1	.653143	.388037	0	
Y(7,7,2)	2.09096	1.84777	1.75632	1.41421	1.29129	.870951	.765367	.342412	0	
Y(10,3,3)	2.09334	1.89764	1.65526	1.41421	1.32808	1	.674863	.239929	0	
Y(9,4,3)	2.101	1.87938	1.61803	1.53209	1.25929	1	.618033	.347295	0	
Y(8,5,3)	2.10445	1.85617	1.67551	1.50102	1.2366	1	.680549	.270459	0	
Y(7,6,3)	2.10579	1.83214	1.73204	1.41419	1.30357	1	.643456	.309005	0	
Y(8,4,4)	2.10814	1.85529	1.61803	1.5545	1.29464	.922103	.618034	.413327	0	
Y(7,5,4)	2.11115	1.82488	1.69244	1.51286	1.27859	.904307	.706694	.328244	0	
Y(6,6,4)	2.11199	1.80193	1.73205	1.49637	1.24698	1	.548062	.445042	0	
Y(6,5,5)	2.11367	1.78556	1.73205	1.5449	1.18147	1	.687435	.29866	0	

Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8			GRAPH
Y(15,1,1)	1.99133	1.9237	1.79052	1.59604	1.3474	1.05286	.722484	.367539	0	0	
Y(14,2,1)	2.01857	1.9319	1.78423	1.57046	1.30178	1	.738979	.517642	.18388		
Y(13,3,1)	2.02368	1.87939	1.65458	1.53209	1.23399	1	.802699	.347296	0	0	
Y(12,4,1)	2.04554	1.02071	1.74465	1.5417	1.36119	1.05963	.688348	.462171	.206202		
Y(11,5,1)	2.05059	1.91039	1.73205	1.58853	1.33659	1	.792579	.455055	0	0	
Y(10,6,1)	2.0533	1.89811	1.74888	1.59334	1.28806	1.078	.750598	.381639	.231494		
Y(9,7,1)	2.05464	1.88563	1.77779	1.55855	1.32005	1.06803	.702451	.506545	0	0	
Y(8,8,1)	2.05504	1.87938	1.79089	1.53209	1.36293	1	.806771	.347297	.247108		
Y(13,2,2)	2.05779	1.93175	1.76717	1.52742	1.22636	1	.881531	.517636	.166526		
Y(12,3,2)	2.07619	1.92436	1.74041	1.48215	1.29036	1.1098	.772807	.482883	.181556		

Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8			GRAPH
Y(11,4,2)	2.08468	1.9136	1.71103	1.52309	1.34743	1	.777305	.544437	.168696		
Y(10,5,2)	2.08881	1.89966	1.70426	1.58468	1.26942	1	.859939	.479173	.178392		
Y(9,6,2)	2.09078	1.88309	1.74483	1.55595	1.21559	1.12736	.743245	.535475	.171534		
Y(8,7,2)	2.09158	1.86778	1.77969	1.49467	.33113	1	.826152	.50037	.17488		
Y(11,3,3)	2.09343	1.9142	1.70533	1.41421	1.41421	1.09756	.805226	.438056	0	0	
Y(10,4,3)	2.10115	1.90014	1.66828	1.54645	1.33967	1.08659	.771827	.435096	.198607		
Y(9,5,3)	2.10471	1.88211	1.68767	1.57015	1.28227	1.10625	.788022	.474498	0	0	
Y(8,6,3)	2.10628	1.86109	1.74515	1.50447	1.31749	1.10183	.799862	.395774	211437	0	
Y(7,7,3)	2.10672	1.84776	1.7716	1.41421	1.41421	1.08616	.765367	.493356	0	0	
Y(9,4,4)	2.10838	1.88186	1.61803	1.61803	1.35774	1.06753	.618034	.618034	.173886		

Tree	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8			Graph
Y(8,5,4)	2.11159	1.85846	1.6959	1.56568	1.35988	1	.816189	.456654	.189339	0	
Y(7,6,4)	2.11283	1.83298	1.75045	1.52841	1.34329	1.07687	.717157	.514509	.180837		
Y(7,5,5)	2.11449	1.82815	1.73204	1.5821	1.31778	1	.841373	.489089	0	0	
Y(6,6,5)	2.11529	1.80171	1.76494	1.58876	1.24698	1.11289	.78164	.445042	.193837		

Bibliography

[1] Baker, T.P., A. Kandel, and S. Matt, *Discrete Mathematics for Computer Scientists,* Reston Publishing Company, Inc., Reston (1983).

[2] Beineke, L.W., and R.J. Wilson, On The Eigenvalues of a Graph, *Selected Topics in Graph Theory,* A Subsidiary of Harcourt & Brace Jovanovich, Publishers (1978) pp.307-336.

[3] Brown, J.W., R.F. Verhey, and R.V. Churchill, *Complex Variables and Applications,* McGraw-Hill Book Company, Tokyo (1974).

[4] Collatz, L., and U. Sinogowitz, *Spektren endichc Grafen, Abh. Math. Sem. Univ.,* Hamburg 21 (1957) pp.63-77; MR 19 # 443.

[5] Frobenius, G., Uber Matrizen aus nicht negativen *Elementen, Sitzber. Akad.* Wiss., Berlin (1912) pp.256-477.

[6] Gantmacher, F.R., *Applications of the Theory of Matrices, Vol. II* Interscience, New York and London, (1959); MR 21 # 6372b.

[7] Hall, M.Jr., *Combinatorial Theory,* John Wiley and Sons, Inc., New York, (1967).

[8] Harary, F., *Graph Theory,* Addison-Welsley Publishing Company, Inc., Reading, Mass., (1972)

[9] Harary, F., C. King, A. Mowshowitz and R.C. Rend, *Cospectral Graphs and Digraphs,* London Math. Soc., 3(1971), PP.321-328; MR45 # 3249.

[10] Harary, F., and A.J. Schwenk, *Which Graphs Have Integral Spectra in Graphs and Combinatorics,* Lecture Notes in Mathematics 406 (ed. R.A. Bari and F. Harary), Springer-Verlag, Berlin, Heidelberg, New York, (1974), pp.45-51.

[11] Heilmann, O.J., and E.H. Lieb, *Theory of Monomer-Dimer System,* Comm. Math. P., 25(3), (1973), PP. 190-232.

[12] Lovasz, L., and J. Pelikan, *On The Eigenvalues of Trees,* Periodica Mathematica Hungarica, Vol. 3(1-2), (1973), PP.175-182.

[13] Neumaier, A., *The Second Largest Eigenvalue of a Tree* Linear Algebra and itsd Applications, 46:9-25 (1982), PP.9-25.

[14] Perron, O., *Zur Theorie der Matrizen,* Math. Ann. 64 (1907), PP.248-263.

[15] Sachs, H., *Bezichungen Zwischen den in einem Graph enthaltenen Kreisen seinem charackteristischen polynom* Publ. Math. Debrecen 11 (1964), PP. 119-134; MR30 # 2491.

[16] Schwenk, A.J., *Computing the Characteristic Polynomial of a Graph, in Graphs and and Combinatorics,* Lecture Notes in Mathematics 406 (ed. R.A. Bari and F. Harary), Spring-Verlag, Berlin, Heidelberg, New York, (1974), PP.247-261 ; MR52 # 7972.

[17] Smith, J.H., *Some Properties of the Spectrum of a graph , in Combinatorial Structures and their Applications* (ed. R.K. Guy et al.), Gordon and Breach, New York, (1970) PP.403-406; MR42 # 1702.

[18] Swamy, M.N.S., and K. Thulasiraman, *Graphs, Networks and Algorithms,* John Wiley and Sons, Inc., New York, (1981).